Classification of random walks in Z

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CLASSIFICATION OF RANDOM WALKS IN Z_+^4

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Abstract

We consider networks with 4 queues, identical customers and interactions between nodes. We get necessary and sufficient conditions for their ergodicity and recurrence.

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Classification de marches aléatoires dans Z_+^4

Irina Ignatyuk, Vadim Malyshev*

Résumé

On considère des réseaux comportant quatre files d'attente. Les clients sont identiques et il y a des intéractions entre les files. On obtient les conditions nécessaires et suffisantes d'ergodicité et de récurrence.

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Chapter 1

Results

1.1 General introduction

This work is a natural continuation of the paper [1], where the classification of homogeneous random walks (r.w.) in $\mathbf{Z}_+^3$ was obtained. Dimension 4 differs essentially from dimension 3 in two main aspects :

1. The law of large numbers which governs the deterministic large scale behaviour of three-dimensional random walks is not valid any more. Phenomenon of scattering appears. This implies that together with the stationary probabilities for ergodic induced chains one must also calculate exit boundaries for nonergodic induced chains. The main consequence of this phenomena is that null recurrent random walks are of positive measure the in parameter space. Recall that in $\mathbf{Z}_+^3$ almost all random walks are either ergodic or transient.

2. Even when the law of large numbers holds the underlying dynamical systems can bee of more complicated nature than those in dimension 3. They correspond to and are classified by Poincare - Bendixson theory on S^2 [3]. The necessity of this theory can be readily explained. The r.w. quickly quits the internal part of $\mathbf{Z}_+^4$ and spends all the time close to its boundary. We project the random walk (see [10] for detailed exposition) onto $\frac{1}{16}$ part of the three dimensional sphere, $S^3 \bigcap I\!R_+^4$. The intersection of S^3 with the boundary of $I\!R_+^4$ is homeomorphic to S^2 and has a simplicial structure coinciding with one of the famous Platonian convex polyedra - the tetrahedron. So piecewise smooth dynamical systems on S^2 arise and Poincare - Bendixson theory provides their classification.

Our main results are the following :

1. Necessary and sufficient conditions are obtained for recurrence (or transience) of the random walks in $\mathbf{Z}_+^4$ for almost all (w.r.t. Lebesgue measure) values of parameters.

2. The same for ergodicity with the exception of one type of random walks which have unstable cycle intersecting all faces of the tetrahedron.

The structure of the paper is given in the contents.
As for applications to networks our results complete classification of 4-queues networks with identical customers (see [10]). But the ideas and methods can be also applied to wider classes of network systems.

1.2 Definitions and results

Faces and random walks
We define

$$\mathbb{R}_+^n = \{x = (x^1, \dots, x^n) : x^i \in \mathbb{R}, x^i \geq 0, \dots, x^n \geq 0\}$$

and its open faces

$$\Lambda(I) = \{x = (x^1, \dots, x^n) : x^i > 0 \Leftrightarrow i \in I\},\ I \subseteq \{1, \dots, n\}$$

For $| I |= k$ we call $\Lambda = \Lambda(I)$ a k-dimensional face or simply k-face. $\mathbf{Z}_+^n$ is the subset of points in $\mathbb{R}_+^n$ with integer coordinates, faces of $\mathbf{Z}_+^n$ are $\Lambda \bigcap \mathbf{Z}_+^n$.

We consider discrete time homogeneous Markov chains or random walks (r.w.) $\mathfrak{L}$ with the set of states $\mathbf{Z}_+^n$ and one step transition probabilities $p(x, y), x, y, \in \mathbf{Z}_+^n$.

In the sequel we make some assumptions $A_i, i = 0, 1, \dots$, which are assumed always to be satisfied.

Assumption A_0 :

1. (homogeneity) for any pair x, y belonging to the same face of $\mathbf{Z}_+^n$ and any $z \in \mathbf{Z}^n$

$$p(x, x + z) = p(y, y + z) \tag{1.2.1}$$

   More exactly, if one transition is defined then the other is also defined and they are equal.

2. (boundedness of jumps) for any $x = (x^1, \dots, x^n), y = (y^1, \dots, y^n)$

$$p(x, y) = 0 \ \text{ if } \ \sum_{j=1}^{n} | x^j - y^j |> d$$

   where $d > 0$ is a fixed positive constant.

   From (1.2.1) it follows in fact that $p(x, y) = 0$ if $x^j - y^j > -1$ at least for some i.

If we speak about r.w. or just about a chain in the sequel, it means that this r.w. satisfies the condition A_0 (unless otherwise stated).
Induced chains. For any k-face $\Lambda = \Lambda(i_1, \dots, i_k)$ we consider a Markov chain $\mathfrak{L}_\Lambda$ with the set of states

$$C_\Lambda \cap \mathbf{Z}_+^n = \{x \in I\!R_+^n : x^{i_1} = ... = x^{i_k} = 1\} \cap \mathbf{Z}_+^n$$

and transition probabilities

$$p_\Lambda(x,y) = \sum_z p(x,y,z), \ x,y \in C_\Lambda \cap \mathbf{Z}_+^n$$

where the summation is over all vectors z perpendicular to C_Λ.

We call $\mathfrak{L}_\Lambda$ an induced chain.

Note that if $k = n$ then the induced chain is trivial (it has one-point state space).

Note also that any $\mathfrak{L}_\Lambda$ is a r.w. in $\mathbf{Z}_+^{n-\dim\Lambda}$, where $\dim \Lambda(i_1, ..., i_k) = k$.

Let us consider a pair Λ, Λ' with $\Lambda \subset \overline{\Lambda'}$ (the closure of Λ'). Note then that the chain $\mathfrak{L}_{\Lambda'}$ is the induced chain for the r.w. $\mathfrak{L}_\Lambda$, under the natural identification.

Ergodic faces and the second vector field.

Assumption A_1 : The Markov chain $\mathfrak{L}$ is irreducible and aperiodic.
Assumption A_2 : Any induced chain $\mathfrak{L}_\Lambda$ is irreducible and aperiodic.
A face Λ is called ergodic if the induced chain $\mathcal{L}_\Lambda$ is ergodic. For any ergodic face Λ we define the vectors :

$$M_\Lambda = \sum_{y \in \mathbf{Z}_+^n} p(x,y)(y-x), \ x \in \Lambda \bigcap \mathbf{Z}_+^n$$

(it is simply the mean jump vector from an arbitrary point $x \in \Lambda \bigcap \mathbf{Z}_+^n$);

$$V_\Lambda = M_\Lambda, \text{ if } \Lambda = \Lambda(1,2,...,n),$$

and

$$V_\Lambda = \sum_{x \in \mathbf{Z}_+^n \cap C_\Lambda} \pi_\Lambda(x)\, P_\Lambda\Big(\sum_{y \in \mathbf{Z}_+^n} p(x,y)(y-x)\Big), \tag{1.2.2}$$

if $\Lambda \neq \Lambda(1,2,...,n)$, where P_Λ is the ortogonal projection onto Λ and $\pi_\Lambda(x), x \in \mathbf{Z}_+^n \cap C_\Lambda$, are stationary probabilities for the induced chain $\mathfrak{L}_\Lambda$.
We call M_Λ the first vector field and V_Λ the second vector field.

Relativisation. We can take $\mathfrak{L}_\Lambda$ instead of $\mathfrak{L}$ and repeat all the above definitions for $\mathfrak{L}_\Lambda$. Then the corresponding vectors can be denoted by $M_{\Lambda'}^\Lambda$, $V_{\Lambda'}^\Lambda$, for any pair $\Lambda \subset \overline{\Lambda'}$. It is easy to see that

$$\begin{cases} M_{\Lambda'}^\Lambda &= (1-P_\Lambda)M_{\Lambda'}\,, \\ V_{\Lambda'}^\Lambda &= (1-P_\Lambda)V_{\Lambda'}\,. \end{cases} \tag{1.2.3}$$

Assumption A_3 : Any vector V_Λ has exactly $k = \dim\Lambda$ nonzero components.

Theorem 1.2.1 *If there exists an ergodic face Λ such that all nonzero components of V_Λ are positive then $\mathfrak{L}$ is transient.*

Proof of this theorem is similar to the corresponding proof in [1]. See also [10].

Ingoing, outgoing and neutral faces.

Let $\Lambda_1 \subset \overline{\Lambda}$ and Λ be ergodic. There are three possibilities for the direction of V_Λ with respect to Λ_1. We say that Λ is an ingoing (outgoing) face for Λ_1 if all nonzero coordinates of $V_\Lambda^{\Lambda_1}$ are negative (positive). Otherwise we say that Λ is neutral for Λ_1.

It is convenient now to define vectors V_Λ also for nonergodic faces Λ. We do it for $n = 4$ (the case we only need).

Let Λ be a nonergodic face. If $\dim\Lambda = 2$ or 3 then there exists at least one outgoing face Λ_1 for Λ. For $\dim\Lambda = 3$ it is trivial and such outgoing face is unique, for $\dim\Lambda = 2$ this follows from [1]. We put in this case

$$V_\Lambda = V_{\Lambda_1} \tag{1.2.4}$$

for arbitrarily chosen outgoing Λ_1. If $\dim\Lambda = 1$ we use also (4) if an outgoing face Λ_1 for Λ is unique and we put $V_\Lambda = 0$ otherwise.
For "relativised" vectors $V_\Lambda^{\Lambda'}$ we use (3) also for nonergodic Λ.
The central notion of all the following is the second vector field $V(x)$ on $I\!R_+^N$. We put

$$\begin{aligned} V(0) &= 0 \text{ and },\\ V(x) &= V_\Lambda,\ x \in \Lambda \subset I\!R_+^N . \end{aligned}$$

For any induced chain $\mathfrak{L}_{\Lambda_1}$ we define the second vector field $V^{\Lambda_1}(x)$ for $x \in C_{\Lambda_1}$ setting

$$\begin{aligned} V^{\Lambda_1}(x) &= 0,\ x \in C_{\Lambda_1} \cap \Lambda_1 ;\\ V^{\Lambda_1}(x) &= V_\Lambda^{\Lambda_1},\ x \in C_{\Lambda_1} \cap \Lambda . \end{aligned}$$

Let us consider the system of differential equations

$$\begin{aligned} &\frac{d}{d_+t}\, T^t(x) = V(T^t(x)),\ t \in I\!R_+,\\ &T^0(x) = x,\ x \in I\!R_+^4 \end{aligned} \tag{1.2.5}$$

where $\frac{d}{d_+t}$ is the right derivative. It is clear that for any $x \in I\!R_+^4$ there exists a unique solution $T^t(x)$, $t \in I\!R_+$, of (1.2.5), and for any $t, s \in I\!R_+$, $T^{t+s}(x) = T^t(T^s(x))$.
So the vector field $V(x)$, $x \in I\!R_+^4$, generates the one parameter semigroup

$$T^t : I\!R_+^4 \to I\!R_+^4,\ t \in I\!R_+ .$$

Similarly for any induced chain the vector field $V^\Lambda(x)$, $x \in C_\Lambda$, defines the one parameter semigroup $T_\Lambda^t : C_\Lambda \to C_\Lambda$, where $T_\Lambda^t(x)$ is defined by

$$\frac{d}{d_+t}\, T_\Lambda^t(x) = V_\Lambda(T^t(x)),\ t \in I\!R_+ \tag{1.2.6}$$

$$T^0_\Lambda(x) = x, x \in C_\Lambda$$

Assumption A_4. For any $x \in I\!R^4_+$, such that $V(x) \neq 0$, $T^t(x) \neq x$ for all $t \in I\!R_+, t \neq 0$.
Assumption A_5. For any Λ and any $x \in C_\Lambda$, such that $V^\Lambda(x) \neq 0, T^t_\Lambda(x) \neq x$ for all $t > 0$.

It is worth notice that for Λ of dimension 3 or 2 the assumption A_5 follows from assumption A_3. For one-dimensional faces A_3 follows from the results of [1].

The following **scaling property**

$$k\, T^t(x) = T^{kt}(kx) \tag{1.2.7}$$

holds for all $x \in I\!R^4_+$, $t \in I\!R_+$, $k \in I\!R_+$.
It is very convinient now to introduce the spherical coordinates : for all $x \in I\!R^4_+ - \{0\}$ define a pair $(\varphi(x), r(x))$, with

$$\varphi(x) = \frac{x}{\| x \|},\ r(x) = \| x \| = (\sum_{j=1}^{4}(x^j)^2)^{\frac{1}{2}}\ .$$

Put for any $x \in I\!R^4_+$

$$t_0(x) = \ \inf\{t \in I\!R_+\ :\ T^t(x) = 0\} \text{ if such } t \text{ exists}$$

$$\text{and } t_0(x) = \infty \text{ otherwise .}$$

From the scaling property (1.2.7) it easily follows that $t_0(x) = r(x)t_0(\varphi(x))$.

For any $t \in I\!R_+$ let us consider the map $G_t\ :\ I\!R^4_+ \to I\!R^4_+$, where $G_t(x) = T^{r(x)t}(x)$, $x \in I\!R^4_+$.

By scaling property, for any $x \in I\!R^4_+$, and any $t \in I\!R_+, 0 \leq t < t_0(\varphi(x))$, the ray emanating from the origin and passing through x, is transformed by G_t into the ray emanating from the origin and passing through $T^t(\varphi(x))$. Moreover for any $y \in I\!R^4_+$, such that $\varphi(x) = \varphi(y)$, we have

$$\frac{r(G_t(y))}{r(y)} = \frac{r(G_t(x))}{r(x)}$$

Now we shall construct on

$$S^4_+ = \{x \in I\!R^4_+\ : \| x \| = 1\}$$

the one-parameter semigroup $T^t_s\ :\ S^4_+ \to S^4_+$, $t \in I\!R_+$, so that the trajectories $\{T^t_s(x),\ t \in I\!R_+\}$ and $\{\varphi(G_t(x)),\ t \in I\!R_+,\ t < t_0\}$ coincide for any $x \in S^4_+$. For this, we define for any point $x \in S^4_+$ the vector

$$V_S(x) = V(x) - (x, V(x))x\ ,$$

where $(x,y) = \sum_{j=1}^{4} x^j y^j$, $x = (x^1,...,x^4)$, $y = (y^1,...,y^4) \in I\!R^4$. Consider the following system of differential equations

$$\frac{d}{dt}T_S^t(x) = V_S(T_S^t(x)), \quad t \in R_+ ,$$

$$T_S^0(x) = x, \; x \in S_+^4 . \tag{1.2.8}$$

It is easy to see that for any $x \in S_+^4$ there exists the unique solution of the system (1.2.8) $T_S^t(x)$, $t \in I\!R_+$, and for any $t, s \in I\!R_+$

$$T_S^t(T_S^s(x)) = T_S^{t+s}(x)$$

Moreover it is easy to see that the trajectories $\{T_S^t(x),\ t \in I\!R_+\}$ and $\{\varphi(G_t(x)),\ t \in I\!R_+,\ t < t_0\}$ coincide for any $x \in S_+^4$ .This means that there exists an increasing continuous function $\alpha_x : [0, t_0(x)] \to I\!R_+$, such that $\alpha_x(t) \uparrow \infty$ as $t \to t_0(x)$, and

$$\varphi(G_t(x)) = T_S^{\alpha_x(t)}(\varphi(x)) \tag{1.2.9}$$

Let us consider the trajectories $\{T_S^t(x),\ t \in I\!R_+\}$.

Critical points

A point $x_0 \in S_+^4$ is called a critical point of the vector field $V_S(x), x \in S_+^4$, if $V_S(x_0) = 0$.

Otherwise this point is said to be a regular point of the vector field $V_S(x)$, $x \in S_+^4$.

Note that any face Λ has at most one critical point. So the vector field $V_S(x)$, $x \in S_+^4$, has a finite number of critical points, each of these points being isolated.

A critical point $x_0 \in S_+^4$ is said to be stable if the following two conditions are satisfied :

a) for any $\epsilon > 0$ there exists $\delta > 0$, such that for any $y \in S_+^4$, for which $\| y - x_0 \| < \delta$, the inequality $\| T_S^t(y) - x_0 \| < \epsilon$ holds for all $t \in R_+$;

b) there exists $\delta > 0$ such that for any $y \in S_+^4$, for which $\| y - x_0 \| < \delta$,

$$T_S^t(y) \to x_0 \text{ as } t \to \infty .$$

Theorem 1.2.2 *Let there exist a stable critical point $x_0 \in S_+^4$ for which*

$$(V(x_0), x_0) > 0 .$$

Then the Markov chain $\mathfrak{L}$ is transient.

In fact the theorem 1.2.2. is a reformulation of the theorem 1.2.1. To show this note that any nonergodic k-face with $k > 1$ has no critical points. Note also that any ergodic k-face Λ with $k > 1$ has a stable critical point if and only if the vector V_Λ has k positive coordinates. It will be shown in the §1.3 that any 1-face Λ is ergodic if and only if the point $x_0 \in \Lambda \cap S_+^4$ is a stable critical point.

Let us now consider the case when for any ergodic face Λ, and in particular for $\Lambda = \Lambda(1,2,3,4)$, the vector V_Λ has at least one negative component.

Let us denote by $\mathfrak{C}$ the union of the all faces Λ for which $V_\Lambda = V_{\Lambda(1,2,3,4)}$. It is easy to see that for any $x \in I\!R^4_+ \backslash \mathfrak{C}$ $\quad T^t(x) \in I\!R^4_+ \backslash \mathfrak{C}$ for all $t \in I\!R_+$, and moreover for any $x \in \mathfrak{C}$ there exists $t_x \in I\!R_+$ such that $T^t(x) \in I\!R^4_+ \backslash \mathfrak{C}$ for all $t \geq t_x$.

For the semigroup $T^t_S, t \in I\!R_+$, we get from this $T^t_S(x) \in S = S^4_+ \backslash \mathfrak{C}$ for any $x \in S$ and for any $t \in I\!R_+$, and moreover for any $x \in S^4_+ \bigcap \mathfrak{C}$ there exists $t_x \in I\!R_+$ such that $T^t_S(x) \in S$ for all $t \geq t_x$.

Let us consider the restriction of the semigroup T^t_S, $t \in I\!R_+$, onto S.

The set $S^4_+ \backslash \Lambda(1,2,3,4)$ is topologically equivalent to a 2-dimensional sphere and we shall show that for the semigroup T^t_S, $t \in I\!R_+$, the analog of the Poincare-Bendixson theory holds.

We call $x_0 \in \overline{S}$ a limit point of the trajectory $\{T^t_S(x),\ t \in I\!R_+\}$ if there exists an increasing sequence $\{t_n\}_{n \in \mathbf{Z}_+} \subset I\!R_+$ such that $t_n \to \infty$ and $T^{t_n}_S(x) \to x_0$ as $n \to \infty$. The set of all limit points of the trajectory $\{T^t_S(x), t \in I\!R_+\}$ we denote by $L(x)$.

Theorem 1.2.3 *For any $x \in S$ the set $L(x)$ is a nonempty connected closed subset of S and either*

a) $L(x)$ consists of a single point, which is a critical point of vector field $V_S(y)$, $y \in S^4_+$, or

b) $L(x)$ is a periodic trajectory ; or

c) $L(x)$ consists of a finite number of critical points and a set of trajectories, each of which tends to one of these critical points as $t \to \infty$.

This theorem will be proved in the §1.3.
Let us note that for any periodic trajectory

$$\gamma = \{T^t_S(x), t \in I\!R_+\}$$

the set $S^4_+ \backslash \Lambda(1,2,3,4) \backslash \gamma$ consists of two sets $S_1(\gamma)$ and $S_2(\gamma)$ such that :

$$S_1(\gamma) \cap S_2(\gamma) = \emptyset\ .$$

Each of the sets $S_1(\gamma)$ and $S_2(\gamma)$ is connected and for any $x \in \overline{S_i(\gamma)}$ $(i = 1,2)$

$$T^t_S(x) \in \overline{S_i(\gamma)} \text{ for all } t \in I\!R_+.$$

A periodic trajectory γ of the semigroup T^t_S, $t \in I\!R_+$, is said to be stable from the side $S_1(\gamma)$ (from the side $S_2(\gamma)$) if for any $\epsilon > 0$ there exists $\delta > 0$, such that for any $y \in S_1(\gamma)$ (for any $y \in S_2(\gamma)$), for which

$$\rho(y, z) = \inf_{z \in \gamma} \| y - z \| < \delta,$$

the inequality

$$\rho(T_S^t(y), \gamma) < \epsilon$$

holds for all $t \in I\!R_+$.

A periodic trajectory γ is said to be stable iff it is stable from the both sides.
For any periodic trajectory $\gamma = \{T_S^t(x),\ t \in I\!R_+\}$ let us consider the value

$$\mathcal{L}_\gamma = log \parallel T^{t_\gamma(x)}(x) \parallel ,$$

where

$$t_\gamma(x) = \inf \{t \in I\!R_+ : \varphi(T^t(x)) = x, t \neq 0\}.$$

Note that by the assumption A_4 for any periodic trajectory γ of the semigroup $T_S^t,\ t \in I\!R_+$,

$$\mathcal{L}_\gamma \neq 0 .$$

Let us note that by the assumption A_0 (with the fixed coonstant d) a random walk in $\mathbf{Z}_+^4$ is completely defined by the finite number of parameters $p(x, y)$.Moreover the parameter space $\mathcal{P}_d$ is a product of 2^N simplexes in finite dimensional euclidean space. Any point $\theta \in \mathcal{P}_d$ corresponds to some r.w. in $\mathbf{Z}_+^4$.

Note that all of the assumptions $A_0 - A_5$ are satisfied for allmost all (w.r.t. Lebesgue measure) values of parameters. Together with $A_0 - A_5$ we shall some also :

Assumption A_6 : The semigroup $T_S^t,\ t \in I\!R_+$, has a finite number of periodic trajectories

Assumption A_7 : Any periodic trajectory of the semigroup $T_S^t,\ t \in I\!R_+$, is either stable from both sides or unstable from both sides.

Assumption A_8 : Any periodic trajectory of the semigroup $T_S^t,\ t \in I\!R_+$, enters a 2-face at the time of quitting any ergodic 3-face.

We shall show in §1.4 that each of these additional assumptions is satisfied on the set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$.

Theorem 1.2.4 .(Transience for stable cycles.)

Let there exist a stable periodic trajectory γ of the semigroup $T_S^t, t \in I\!R_+$, such that $\mathcal{L}_\gamma > 0$.
Then the r.w. $\mathfrak{L}$ is transient.

Let us now consider cases when the conditions of the theorem 1.2.2. and 1.2.4. are not satisfied.

Ingoing and outgoing separatrices

Let z_0 be a critical point of the vector field $V_S(x), x \in S_+^4$, and let there exist sequences $\{x_n\}_{n \in \mathbb{Z}_+} \subset S$ and $\{t_n\}_{n \in \mathbb{Z}_+} \subset \boldsymbol{R}_+$ such that for any $n \in \mathbb{Z}_+$

$$x_n = T_S^{t_n}(x_{n+1}),$$

and

$$\sup_{t \in [0, t_n]} \| T_S^t(x_{n+1}) - z_0 \| \to 0 \text{ as } n \to \infty .$$

Then the union $\gamma = \bigcup_{n=0}^{\infty} \{T_S^t(x_n),\ t \in \boldsymbol{R}_+\}$ is said to be a separatrix outgoing from the point z_0.

Let z_0 be a critical point of the vector field $V_S(x), x \in S_+^4$. Let there exist sequences $\{x_n\}_{n \in \mathbb{Z}_+} \subset S_+^4$ and $\{t_n\}_{n \in \mathbb{Z}_+}$ such that for any $n \in \mathbb{Z}_+$ $x_n = T_S^{t_n}(x_{n+1})$ and $T_S^t(x_n) \to z_0$ as $t \to \infty$. Then the union

$$\gamma = \bigcap_{n=0}^{\infty} \{T_S^t(x_n),\ t \in \boldsymbol{R}_+\}$$

is said to be a separatrix ingoing to the point z_0.

Associated graph

Let us consider all nonstable critical points of the vector field $V_S(x), x \in S_+^4$, which belong to a 1-face having at least two outgoing 2-faces. We denote the set of this points by $\mathcal{X}$.

It is convenient to introduce a directed graph $\mathcal{G}$ having the set $\mathcal{X}$ as a set of vertices, and the set of all separatrices outgoing from the points of $\mathcal{X}$ and ingoing to the points of $\mathcal{X}$ as a set of edges. (For any $x, y \in \mathcal{X}$ a separatrix outgoing from $\mathcal{X}$ and ingoing to y is an edge from x to y in the graph $\mathcal{G}$).

This directed graph $\mathcal{G}$ is said to be an associated graph for the r.w. $\mathfrak{L}$.

Assumption A_9 : For any $x \in \mathcal{X}$, any separatrix outgoing from x enters a 2-face at the time of quitting any 3-face.

This assumption, as well as the assumptions A_6, A_7 and A_8, is satisfied on the set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$. This will be shown in §1.4.

Theorem 1.2.5 .(Ergodicity for weakly acyclic case.)

Let the associated graph $\mathcal{G}$ has no cycles and the following conditions hold

a) for any stable critical point x_0 of the vector field $V_S(x), x \in S_+^4$,

$$(V(x_0), x_0) < 0 ;$$

b) for any periodic trajectory γ of the semigroup $T_S^t, t \in \boldsymbol{R}_+$,

$$\mathcal{L}_\gamma < 0 .$$

Then the Markov chain L is ergodic.

Theorem 1.2.6 .(Null recurrence for unstable cycles.)

Let the associated graph $\mathcal{G}$ has no cycles and the following conditions hold

a) for any stable critical point $x_0 \in S_+^4$

$$(V(x_0), x_0) < 0 \ ;$$

b) for any stable periodic trajectory γ

$$\mathcal{L}_\gamma < 0 \ ;$$

c) there exists an unstable periodic trajectory γ for which

$$\mathcal{L}_\gamma > 0 \ .$$

Then the r.w. $\mathfrak{L}$ is null recurrent.

We shall not prove this theorem here.
Let us now consider the case when the associated graph $\mathcal{G}$ has a cycle.

Scattering probabilities

Let Λ be a nonergodic face and Λ_1 be its outgoing fase.

Let us consider the r.w. $\xi_x^\Lambda(t)$ starting at the point $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ and corresponding to the induced Markov chain $\mathfrak{L}_\Lambda$.

For the face Λ_1, let us consider the union of all faces Λ' such that $\Lambda_1 \subset \overline{\Lambda'}$. We shall denote it by $Q(\Lambda_1)$.
The probability

$$g_\Lambda(x, \Lambda_1) = P\{\bigcup_{t \geq 0} \{\xi_x^\Lambda(\tau) \in Q(\Lambda_1) \forall \tau \geq t\}\} \tag{1.2.10}$$

may be called a probability to go to infinity from the point x along the face Λ_1.

As it will be shown in §1.5 for any nonergodic face Λ having at least one outgoing face, the following equality holds

$$\sum_{\Lambda_1} g_\Lambda(x, \Lambda_i) = 1 \ ,$$

for all $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$, where the summation is over all outgoing faces Λ_i of Λ.

Let Λ be a 1-face having outgoing faces and having also some ingoing face Λ_1. For any outgoing face Λ_2 and for any $x \in C_\Lambda$ let us consider the following limit

$$g(\Lambda_1, \Lambda_2) = \lim_{n \to \infty} g_\Lambda(x + ne_{\Lambda,\Lambda_1} \ ; \ \Lambda_2) \tag{1.2.11}$$

where $e_{\Lambda,\Lambda_1} \in \overline{\Lambda}_1 \bigcap C_\Lambda$ is a unit vector perpendicular to Λ.

Assumption A_{10}. For any 1-face Λ having ingoing and outgoing faces, for any face Λ_1 ingoing to Λ, and for any face Λ_2 outgoing from Λ the limit (1.2.11) exists and does not depend on $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$.

We shall show in §1.5 that the assumption A_{10} is satisfied for almost all (with respect to Lebesgue measure) values of parameters.

Associated Markov chain

Let us consider, for any $x \in \mathcal{X}$, the set $W_+(x)$ of all separatrices outgoing from x, and the set $W_-(x)$ of all separatrices ingoing to x.Put

$$W_+ = \bigcup_{x \in \mathcal{X}} W_+(x), \; W_- = \bigcup_{x \in \mathcal{X}} W_-(x) \, .$$

We define an associated Markov chain $\mathfrak{A}$ having the space of states W_+. Let us define one-step transition probabilities of the chain $\mathfrak{A}$.
For all $\gamma \in W_+ \backslash W_-$ we set

$$p_{\mathfrak{A}}(\gamma, \gamma') = \begin{cases} 1, & \text{if } \gamma = \gamma' \, ; \\ 0, & \text{otherwise} \, . \end{cases}$$

Let $\gamma \in W_+ \bigcap W_-(x)$ for some $x \in \mathcal{X}$. Then for any $\gamma' \in W_+ \backslash W_+(x)$ we set

$$p_{\mathfrak{A}}(\gamma, \gamma') = 0 \, .$$

To define $p_{\mathfrak{A}}(\gamma, \gamma')$ for $\gamma' \in W_+(x)$ let us consider the 1-face Λ, containing the point x. Note that the ingoing separatrix γ, by the assumption A_9, goes to x along some ingoing to Λ 2-face Λ_1, and the outgoing separatrix γ' goes from x along some outgoing from Λ 2-face Λ_2. This means that there exists $z \in \gamma$ such that $\{T_S^t(z), t \in I\!\!R_+\} \subset \Lambda_1$ and $\gamma' \bigcap \Lambda_2 = \Lambda_2 \bigcap S_+^4$.
So we set

$$p_{\mathfrak{A}}(\gamma, \gamma') = g(\Lambda_1, \Lambda_2) \, .$$

Let us define on the set of states of the associated Markov chain $\mathfrak{A}$ the following function K_γ, $\gamma \in W_+$:
for all $\gamma \in W_+ \backslash W_-$ we set

$$K_\gamma = \frac{1}{2} \, .$$

For $\gamma \in W_+ \bigcap W_-$ let us consider $x \in \mathcal{X}$, for which $\gamma \in W_+(x)$, and $x' \in \mathcal{X}$ for which $\gamma \in W_-(x')$. It is easy to see that for any $y \in \gamma$

$$t(y) = \inf \{ t \in I\!\!R_+ : \varphi(T^t(y)) = x' \} < \infty$$

and there exists a limit

$$\lim_{y \to x, y \in \gamma} \| \, T^{t(y)}(y) \, \|$$

So we put

$$K_\gamma = \lim_{y \to x, y \in \gamma} \| \, T^{t(y)}(y) \, \| \; .$$

Let $\gamma_0, ..., \gamma_n, ...$ be random states of $\mathfrak{A}$ with $\gamma_0 \in W_+$.

Proposition 1.2.7 *The following limit*

$$\lim_{n\to\infty} \frac{1}{n}\, log\,(E\prod_{j=1}^{n} K_{\gamma_j}) = \mathcal{M}_{\gamma_0}$$

exists. Moreover let $W[\gamma_0]$ be the set of all states which are accessible from the state γ_0, then

$$\mathcal{M}_{\gamma_0} = \; log\; \lambda(\gamma_0), \tag{1.2.12}$$

where $\lambda(\gamma_0)$ is a maximal eigenvalue of the matrix, with matrix elements

$$A(\gamma,\gamma') = p_{\mathfrak{A}}(\gamma,\gamma')\sqrt{K_\gamma K_{\gamma'}},\ \gamma,\gamma' \in W[\gamma_0]\ .$$

Proof : Let us consider the matrix A with matrix elements

$$A(\gamma,\gamma') = p_{\mathfrak{A}}(\gamma,\gamma')\sqrt{K_\gamma K_{\gamma'}},\ \gamma,\gamma' \in W_+\ .$$

One can easily show that for any $\gamma_0 \in W_+$

$$E(\prod_{j=1}^{n} K_{\gamma_j}) = (l(\gamma_0), A^n\ l_1)\ , \tag{1.2.13}$$

where the vectors $l(\gamma_0), l_1 \in I\!R^{|W_+|}$ are defined in the following way :

$$l^\gamma(\gamma_0) = \begin{cases} (K_\gamma)^{-\frac{1}{2}} & \text{if } \gamma = \gamma_0\ , \\ 0 & \text{otherwise}\ ; \end{cases}$$

and

$$l_1^\gamma = (K_\gamma)^{\frac{1}{2}} \text{ for all } \gamma \in W_+\ .$$

Let $V(\gamma_0)$ be a subspace of $I\!R^{|W_+|}$ generated by the vectors $l(\gamma_0),\ A^T l(\gamma_0), ..., (A^T)^n l(\gamma_0), ...,$ where A^T is the transpose matrix of A. $V(\gamma_0)$ is an eigen subspace of A^T.
As it follows from Perron-Frobenious theorem

$$\frac{1}{n}\ \log\ (l(\gamma_0),\ A^n l_1) \to \lambda(\gamma_0) \text{ as } n \to \infty,$$

where $\lambda(\gamma_0)$ is the maximal eigenvalue of the restriction of the matrix A^T on the eigen subspace $V(\gamma_0)$.

To prove now the proposition 1.2.7 it is sufficient to note that $V(\gamma_0)$ is the set of all vectors $\{l^\gamma\}_{\gamma\in W_+}$ such that $l^\gamma = 0$ for all $\gamma \notin W[\gamma_0]$.

Proposition 1.2.7 is proved.

Let

$$\mathcal{M}_{\mathfrak{A}} = \max_{\gamma\in W_+} \mathcal{M}_\gamma .$$

Assumption A_{11} : $\mathcal{M}_{\mathfrak{A}} \neq 0$.

Theorem 1.2.8 .(Nonergodicity for essential scattering.)

Let $\mathcal{M}_{\mathfrak{A}} > 0$. Then the r.w. $\mathfrak{L}$ is nonergodic.

Theorem 1.2.9 .(Ergodicity for essential scattering.)

Let the following conditions are satisfied

a) for any stable critical point $x_0 \in S_+^4$

$$(V_n(x_0), x_0) < 0 \ ;$$

b) for any periodic trajectory γ

$$\mathcal{L}_\gamma < 0 \ ;$$

c) $\mathcal{M}_{\mathfrak{A}} < 0$.

Then the r.w. $\mathfrak{L}$ is ergodic.

Let $W^1, ..., W^N$ be irreducible classes of essential states of the chain $\mathfrak{A}$.
By the ergodic theorem for any $j = 1, ..., N$ and for any $\gamma \in W^j$ the following limit exists

$$\lim_{n\to\infty} \frac{1}{n} E(\log \prod_{j=1}^{n} K_{\gamma j}) \doteq \mathcal{L}_{W^j} \ , \qquad (1.2.14)$$

and

$$\mathcal{L}_{W^j} = \sum_{\gamma \in W^j} \pi_{\mathfrak{A}}(\gamma) \log K_\gamma \ , \qquad (1.2.15)$$

where $\pi_{\mathfrak{A}}(\gamma), \gamma \in W^j$, are stationary probabilities of the Markov chain $\mathfrak{A}$.
By Iensen's inequality for any $j = 1, ..., N$, and for any $\gamma \in W^j$

$$\mathcal{L}_{W^j} \le \mathcal{M}_\gamma. \qquad (1.2.16)$$

Let

$$\mathcal{L}_{\mathfrak{A}} = \max_{j=1,...,N} \mathcal{L}_{W^j}$$

Assumption A_{12} : $\mathcal{L}_{\mathfrak{A}} \neq 0$.

Theorem 1.2.10 .(Transience for essential scattering.)
Let $\mathcal{L}_{\mathfrak{A}} > 0$.
Then the r.w. $\mathfrak{L}$ is transient.

Theorem 1.2.11 .(Recurrence for essential scattering.)
Let the following conditions to be satisfied :

a) for any stable critical point $x_0 \in S^4_+$

$$(V_S(x_0), x_0) < 0 \; ;$$

b) for any stable periodic trajectory γ

$$\mathcal{L}_\gamma < 0 \; ;$$

c) $\mathcal{L}_{\mathfrak{A}} < 0$.

Then the Markov chain $\mathfrak{L}$ *is recurrent.*

1.3 Some results about dynamical systems

Note that in the case when all the components of the vector $V_\Lambda(1,2,3,4)$ are positive, there exists a unique stable critical point $x_0 \in \Lambda(1,2,3,4) \bigcap S^4_+$ and for any $x \in S^4_+$

$$T^t_S(x) \to x_0 \;\text{ as }\; t \to \infty.$$

We shall assume here that the vector $V_{\Lambda(1,2,3,4)}$ has at least one negative component.

Proposition 1.3.1 *.A 1-face* Λ *is ergodic if and only if the point* $x_0 \in \Lambda \bigcap S^4_+$ *is a stable critical point of the vector field* $V(x), x \in S^4_+$.

(It is easy to see that the set $\Lambda \bigcap S^4_+$ for any 1-face Λ consists of a single point)

Proof : Let Λ be a 1-face and let $x_0 \in \Lambda \bigcap S^4_+$. Let us note that $\mathcal{L}_\Lambda$ is isomorphic to a r.w. in $\mathbf{Z}^3_+$ for which all the assumptions of the paper [1] are satisfied. So the chain $\mathfrak{L}_\Lambda$ is ergodic if and only if for any $x \in C_\Lambda$ there exists $t_x(\Lambda) \in \Lambda$, such that $T^t_\Lambda(x) \in \Lambda$ for any $t \geq t_x(\Lambda)$, and

$$\mathcal{H}_1 \rho(x, \Lambda) \leq t_x(\Lambda) \leq \mathcal{H}_2 \rho(x, \Lambda),$$

where $\mathcal{H}_1, \mathcal{H}_2$ are some positive constants.

It follows easily from this remark that if the chain $\mathfrak{L}_\Lambda$ is ergodic, then the point x_0 is a stable isolated critical point.

Let us prove now that from the stability of the point x_0 it follows the ergodicity of the face Λ.

Let x_0 be a stable critical point. By the definition of a stable critical point for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$, such that for any $x \in I\!R^4_+, x \neq 0$, for which $\rho(\varphi(x), \Lambda) < \delta(\epsilon)$, the following inequality holds

$$\rho(\varphi(T^t(x)), \Lambda) < \epsilon$$

for all $t \in I\!R_+$.

Let us choose $\epsilon > 0$ and $\delta(\epsilon) > 0$ such that $V_S(x) \neq 0$ for any $x \in \{z \in S^4_+ : 0 < \| x_0 - z \| < \epsilon\}$ and

$$\{y \in I\!\!R^4_+ : y \neq 0, \rho(\varphi(y), \Lambda) < \epsilon\} \subset Q(\Lambda)$$

where $Q(\Lambda)$ is the union of all faces Λ^* for which $\Lambda \subset \overline{\Lambda^*}$.
Then for any $x \in Q(\Lambda)$, such that $\rho(\varphi(x), \Lambda) < \delta(\epsilon)$ we have $T^t(x) \in \overline{Q(\Lambda)}$ for any $t \in I\!\!R_+$.

Let $x \in Q(\Lambda)$ and $\rho(\varphi(x), \Lambda) < \delta(\epsilon)$. It is easy to see that there exist a 2-dimensional plane $\Pi \subset I\!\!R^4$ and $t_1, t_2 \in I\!\!R_+$, $t_1 < t_2$, such that

$$T^{t_1}(x), T^{t_2}(x) \in \Pi.$$

If $T^{t_2}(x) \in \overline{\Lambda}$, then $T^t(x) \in \overline{\Lambda}$ for any $t \geq t_2$. Let us consider the case when $T^{t_2}(x) \notin \Lambda$. Then by assumptions A_3 and A_5 the straight line, containing the points $T^{t_1}(x)$ and $T^{t_2}(x)$, intersects the straight line, containing the face Λ in some point y.
Let us consider

$$k = \frac{\| y - T^{t_2}(x) \|}{\| y - T^{t_1}(x) \|}$$

and let us consider a sequence $\{t_n\}_{n \in \mathbf{Z}_+} \subset I\!\!R_+$, where $t_{n+1} = t_n + k(t_n - t_{n-1})$ for each $n \in \mathbf{Z}_+, n \geq 1$.

It easily follows from the definition of the vector field $V(x), x \in I\!\!R^4_+$, that all the points

$$y, T^{t_1}(x),\ T^{t_2}(x), ...,\ T^{tn}(x), ...$$

belong to the same straight line, and

$$\| y - T^{t_n}(x) \| = k^{n-1} \| y - T^{t_1}(x) \|$$

for any $n \in \mathbf{Z}_+, n \geq 1$.

Since $T^{t_1}(x) \neq T^{t_2}(x)$, then $k \neq 1$. Moreover it is easy to see that in the case $k > 1$, we have

$$\rho(\varphi(T^{tn}(x)), \Lambda) \not\to 0 \text{ as } n \to \infty\ ,$$

and the point x_0 can not be stable.
So we have $k < 1$. Note that $t_n \to t_x = t_1 + \frac{t_2 - t_1}{1-k}$ as $n \to \infty$, in this case and $T^{tn}(x) \to y$ as $n \to \infty$.
It is obvious that $T^{t_x}(x) = y \in \overline{\Lambda}$, and $T^t(x) \in \overline{\Lambda}$ for any $t \geq t_x$. Moreover it is easy to see that

$$\mathcal{H}_1 \rho(x, \Lambda) \leq t_x \leq \mathcal{H}_2 \rho(x, \Lambda)\ ,$$

where the constants $\mathcal{H}_1, \mathcal{H}_2$ do not depend on x.

So the proposition 1.3.1. is proved.

Proof of the theorem 1.2.3.

Let us consider at first some properties of continuity of the one-parameter semigroup $T_S^t, t \in I\!R_+$.

It is easy to see that for any $x \in S_+^4$ $T_S^t(x)$ is a continuous function of t, and for any $t \in I\!R_+$ the mapping $T_S^t : S_+^4 \to S_+^4$ has in some cases points of discontinuity.

Let us describe all the points of discontinuity of the mappings $T_S^t : S_+^4 \to S_+^4$, $t \in I\!R_+$.

Let $\mathfrak{C}^{(1)}$ be the union of all 1-faces Λ for which we have defined $V_\Lambda = 0$, and let $\mathfrak{C}^{(2)}$ be the union of all 2-faces having two outgoing ergodic faces.

It is easy to see that the mapping $T_S^t : S_+^4 \to S_+^4$ is continuous if and only if

$$\mathfrak{C}^{(1)} \bigcup \mathfrak{C}^{(2)} = \emptyset .$$

Let $t_x^c = \inf \{t \in I\!R_+ : T_S^t \in \mathfrak{C}^{(1)} \bigcup \mathfrak{C}^{(2)}\}, x \in S_+^4$.
It is easy to see that the following proposition holds.

Proposition 1.3.2 *For any $x \in S_+^4$ and for any $t \in I\!R_+$, such that $t < t_x^c$, the mapping $T_S^t : S_+^4 \to S_+^4$ is continuous at the point x.*

Using the propositions 1.3.1 and 1.3.2 let us prove now the theorem 1.2.3.

Proposition 1.3.3 *For any $x \in S_+^4$ $L(x)$ is a non-empty, closed and connected subset of $S \backslash \mathfrak{C}^{(2)}$.*

Proof : For any $x \in S_+^4$ $L(x) \neq \emptyset$ by compactness of the set S_+^4, $L(x)$ is closed by definition, and $L(x)$ is connected by continuity of the function

$$T_S^{\cdot}(x) : I\!R_+ \to S_+^4 .$$

Let us note that $L(x) \subseteq \overline{S}$, where $\overline{S}$ is the closure of S, and $L(x) \bigcap \mathfrak{C}^{(2)} = \emptyset$ by the definition of the vector field $V_S(x), x \in S_+^4$.

To prove now the proposition 1.3.3. it is sufficient to note that $\overline{S} = S$.
Proposition 1.3.3 is proved.

From the proposition 1.3.2 it easily follows that if the set $L(x)$ consists of a single point x_0, then $V_S(x_0) = 0$. Let us also note that if the set $L(x)$ contains a stable isolated critical point x_0, then

$$L(x) = \{x_0\} .$$

To consider the case when $L(x)$ consists of more than one point, we shall use the following proposition.

Proposition 1.3.4 *For any $x \in S_+^4$ and for any $y \in L(x)$*

$$T_S^t(y) \in L(x) \textit{ for all } t \in I\!R_+ .$$

Proof : For $y \in \mathfrak{C}^{(1)} \bigcap L(x)$ this proposition is obvious. Let $y \in L(x) \backslash \mathfrak{C}^{(1)}$. Then for any $t \in I\!R_+$, such that $t < t^c_y$, by continuity of the mapping $T^t_S : S^4_+ \to S^4_+$ in the point y we get $T^t_S(y) \in L(x)$.

For $t = t^c_y$ we have $T^t_S(y) \in L(x)$ by continuity of the function $T^{\cdot}_S(y) : S^4_+ \to S^4_+$ and by closedness of the set $L(x)$.

To consider the case, when $t > t^c_y$, let us note that $y \in S \backslash \mathfrak{C}^{(2)}$ by the proposition 1.3.3, and so we have $T^t_S(y) \in S \backslash \mathfrak{C}^{(2)}$ for any $t \in I\!R_+$, and in particular $T^{t^c_y}_S(y) \in \mathfrak{C}^{(1)}$. Since by definition of the vector field $V_S(z), z \in S^4_+$, we have

$$V_S(z) = 0 \text{ for all } z \in \mathfrak{C}^{(1)},$$

then

$$T^t_S(y) = T^{t^c_y}_S(y) \in L(x) \text{ for any } t \geq t^c_y \, .$$

Proposition 1.3.4 is proved.
From the propositions 1.3.3 and 1.3.4 we get

Proposition 1.3.5 *For any $x \in S^4_+$ and for any $y \in L(x)$*

$$L(y) \subseteq L(x) \, .$$

Proposition 1.3.6 *For any $x \in S^4_+$ and for any $y \in L(x)$ either $L(y)$ consists of a single point, or $L(y)$ is a periodic trajectory.*

Proof: Let $x \in S^4_+$, $y \in L(x)$, and $L(y)$ do not consist of a single point. Then there exists a nonergodic 2-face or 1-face which is intersected by the trajectory $\{T^t_S(y),\ t \in I\!R_+\}$ infinitely often. It is clear that if the trajectory $\{T^t_S(y),\ t \in I\!R_+\}$ intersects infinitely often some nonergodic 1-face, then this trajectory contains a periodic trajectory which is identical with $L(y)$.

Let us consider the case when the trajectory $\{T^t_S(y),\ t \in I\!R_+\}$ intersects infinitely often some nonergodic 2-face Λ. Note first that for any $x \in S^4_+$ and for any nonergodic 2-face Λ either

$$\Lambda \bigcap \{T^t_S(x),\ t \in I\!R_+\} = \emptyset \, ,$$

or the points of the intersection $\Lambda \bigcap \{T^t_S(x),\ t \in I\!R_+\}$ have the same order on the line $\Lambda \bigcap S^4_+$ and on the trajectory $\{T^t_S(x), t \in I\!R_+\}$.

From this it easily follows that if the trajectory $T^t_S(y), t \in I\!R_+$ intersects infinitely often some nonergodic 2-face Λ, then the intersection

$$\{T^t_S(y), t \in I\!R_+\} \bigcap \Lambda$$

consists of a single point. Since the trajectory $\{T^t_S(y), t \in I\!R_+\}$ intersects the face Λ infinitely often in the same point then the trajectory $\{T^t_S(y), t \in I\!R_+\}$ contains a periodic trajectory $\{T^t_S(y), t \geq \tilde{t}\}$, and $\{T^t_S(y), t \geq \tilde{t}\} = L(y)$.
Proposition 1.3.6. is proved.

Using the proposition 1.3.6. one can easily prove the following proposition.

Proposition 1.3.7 *Let for $x \in S^4_+$ there exists $t_1 \in I\!R_+$ such that $T^{t_1}_S(x) \in L(x)$. Then either $V_S(T^{t_1}_S(x)) = 0$ and $L(x) = \{T^{t_1}_S(x)\}$, or $\{T^t_S(x), t \geq t_1\}$ is a periodic trajectory, and $L(x) = \{T^t_S(x), t \geq t_1\}$*

To complete the proof of the theorem 1.2.3 it is now sufficient to prove the following

Proposition 1.3.8 *Let for $x \in S^4_+$ the set $L(x)$ contain a periodic trajectory. Then $L(x)$ is identical with this periodic trajectory.*

Proof : Let $x \in S^4_+, y \in L(x)$ and let the trajectory $\gamma = \{T^t_S(y), t \in I\!R_+\}$ be periodic.

It is easy to see that any periodic trajectory intersects some 2-face.

Let us consider first the case when the trajectory γ intersects some ergodic 2-face Λ. It is easy to see that in this case there exists $t_1 \in I\!R_+$, such that $T^{t_1}_S(x) \in \Lambda \bigcap \gamma$. From this by the proposition 1.3.7. we get

$$L(x) = \gamma \ .$$

To prove the proposition 1.3.8. it is now sufficient to consider the case when the trajectory γ intersects a nonergodic 2-face Λ.

We shall use here the following

Lemma 1.3.9 *Let $x \in S^4_+, t \in I\!R_+$, and $t < t^c_x$. Then for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $y \in S^4_+$, for which $\| x - y \| < \delta$, the following inequality holds*

$$\sup_{\tau \in [0,t]} \| T^\tau_S(x) - T^\tau_S(y) \| < \epsilon \ .$$

The proof of this lemma is standard.

Let $\gamma \bigcap \Lambda = \{x_1\}$, where Λ is a nonergodic 2-face. It is easy to see that in this case the trajectory $\{T^t_S(x), t \in I\!R_+\}$ intersects the face Λ infinitely often. This means that there exists an increasing sequence $\{t_n\}_{n \in \mathbf{Z}_+} \subset I\!R_+$ such that

$$T^{t_n}_S \in \Lambda \text{ for all } n \in \mathbf{Z}_+, \text{ and}$$

$$t_n \to \infty, \text{ as } n \to \infty, \text{ and } T^{t_n}_S(x) \to x_1 \text{ as } n \to \infty \ .$$

Moreover, as it had been noted in the proof of the proposition 1.3.6, the points of intersection $\Lambda \bigcap \{T^t_S(x), t \in I\!R_+\}$ have the same order on the line $\Lambda \bigcap S^4_+$ and on the trajectory $\{T^t_S(x), t \in I\!R_+\}$. So without a loss of generality we can assume that

$$\Lambda \bigcap \{T^t_S(x), t \in I\!R_+\} = \{T^{t_n}_S(x), n \in \mathbf{Z}_+\} \ .$$

Let us consider $t' \in I\!R_+$ such that

$$T^{t'}_S(x_1) = x_1 \ .$$

Then by the lemma 1.3.9 and by convergence $T_S^{t_n}(x) \to x_1$ for any $\epsilon > 0$ there exists $n_\epsilon \in \mathbf{Z}_+$, such that for any $n \geq n_\epsilon, n \in \mathbf{Z}_+$, the following inequality holds

$$\sup_{\tau \in [0,2t']} \| T_S^{t_n+\tau}(x) - T_S^\tau(x_1) \| < \epsilon \,. \tag{1.3.1}$$

It is easy to see that for sufficiently small $\epsilon > 0$

$$\begin{array}{l} t_{n+1} - t_n < 2t' \\ \text{for any } n \geq n_\epsilon, n \in \mathbf{Z}_+ \,. \end{array} \tag{1.3.2}$$

Using (1.3.1.) and (1.3.2.) one can easily show that for any $\epsilon > 0$

$$\rho(L(x), \gamma) < \epsilon \,,$$

and consequently

$$L(x) = \gamma \,.$$

Proposition 1.3.8 is proved.
The theorem 1.2.3 follows from the propositions 1.3.3-1.3.8.
So the theorem 1.2.3 is proved.

1.4 Some geometrical properties of the one-parameter semigroup $T_S^t, t \in I\!R_+$, on a set of complete Lebesgue measure in the parameter space

All the assumptions $A_0 - A_5$ and A_{11}, A_{12} are obviously satisfied on a set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$ of the r.w. $\mathfrak{L}$. We shall show now that all the assumptions $A_6 - A_9$ are also satisfied on a set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$.

Proposition 1.4.1 *On a set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$ of the r.w. $\mathfrak{L}$ the semigroup $T_S^t, t \in I\!R_+$, has no any trajectory which is either a separatrix outgoing from a point of the set $\mathcal{X}$, or a periodic trajectory,and which enters a 1-face at the time of quitting a 3-face.*

Proof :
To prove this proposition let us consider the trajectories of the semigroup $T^t, t \in I\!R_+$. Let a trajectory $\{T^t(x), t \in I\!R_+\}$ cross a 3-face $\Lambda(i,j,k), x \notin \Lambda(i,j,k)$, and let

$$t_1 = \inf \{t \in I\!R_+ : T^t(x) \in \overline{\Lambda(i,j,k)}\} \,,$$

$$x = T^{t_1}(x) \in \overline{\Lambda(i,j)}, \text{ and}$$

$$t_2 = \inf \{t \in I\!R_+ : t > t_1, T^t(x) \notin \Lambda(i,j,k)\} \,,$$

$$y = T^{t_2}(x) \in \overline{\Lambda(j,k)}$$

Then it is easy to see that

$$\begin{pmatrix} y^j \\ y^k \end{pmatrix} = A^j_{ik} \begin{pmatrix} x^j \\ x^i \end{pmatrix} \tag{1.4.1}$$

where

$$A^j_{ik} = \begin{pmatrix} 1 & \dfrac{V^j_{\Lambda(i,j,k)}}{\mid V^i_{\Lambda(i,j,k)} \mid} \\ & \\ 0 & \dfrac{V^k_{\Lambda(i,j,k)}}{\mid V^i_{\Lambda(i,j,k)} \mid} \end{pmatrix}$$

Let us note now that the semigroup $T^t_S, t \in I\!R_+$, has a separatrix outgoing from a point of the set $\mathcal{X}$, which enters some 1-face at the time of quitting a 3-face, if and only if there exist 1-faces Λ_1 and Λ_2 and there exists a trajectory of the semigroup $T^t, t \in I\!R_+$, going from the face Λ_1 to the face Λ_2 and crossing two or three ergodic 3-faces. The same is true for periodic trajectories. So by (1.4.1.) we have

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = B_1 ... B_m \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 2 \leq m \leq 3, \tag{1.4.2}$$

where for any $s = 1, ..., m$

$$B_s = A^j_{ik}$$

for some $i, j, k \in \{1, 2, 3, 4\}$.

From (1.4.2) we get the proposition 1.4.1.

Proposition 1.4.1 is proved.

Proposition 1.4.2 *On a set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$ for any $x \in S^4_+$ either the set of limit points $L(x)$ of the trajectory $\{T^t_S(x), t \in I\!R_+\}$ consists of a single point, or $L(x)$ is a periodic trajectory.*

Proof : Let us first note that, for any $x \in S^4_+$, for which the set $L(x)$ does not consist of a single point, either the trajectory $\{T^t_S(x), t \in I\!R_+\}$ intersects infinitely often some nonergodic 1-face, or for some $t' \in I\!R_+$ the trajectory $\{T^t_S(x), t \geq t'\}$ does not intersect 1-faces and ergodic 2-faces but goes along ergodic 3-faces and crosses nonergodic 2-faces infinitely often.

It is obvious that if the trajectory $\{T^t_S(x), t \in I\!R_+\}$ intersects infinitely often some nonergodic 1-face, then $L(x)$ is a periodic trajectory.

Let us consider the case, when for some $t' \in I\!R_+$, the trajectory $\{T_S^t(x), t \geq t'\}$ does not intersect 1-faces and ergodic 2-faces. Let Λ be a nonergodic 2-face which is intersected infinitely often by the trajectory $\{T_S^t(x), t \geq t'\}$:

$$\Lambda \bigcap \{T_S^t(x), t \in I\!R_+\} = \{T_S^{t_n}(x)\}_{n \in \mathbf{Z}_+},$$

where $t_n \to \infty$ as $n \to \infty$, and

$$0 \leq t_0 < t_1 < t_2 < ... < t_n < ...$$

Since the trajectory $\{T_S^t(x), t \in I\!R_+\}$ intersects the face Λ always in the same direction, the order of the points $T_S^{t_n}(x), n \in \mathbf{Z}_+$, on the trajectory is the same as the order on the line $\Lambda \bigcap S_+^4$. From this it follows that there exists $\hat{x} \in \Lambda \bigcap S_+^4$ such that

$$T_S^{t_n}(x) \to \hat{x} \text{ as } n \to \infty .$$

By the proposition 1.3.6 either $L(\hat{x})$ consists of a single point, or $L(\hat{x})$ is a periodic trajectory.
If $L(\hat{x})$ is a periodic trajectory then, by the proposition 1.3.8, $L(x)$ is identical with this periodic trajectory.
Let us consider the case when $L(\hat{x})$ consists of a single point :

$$L(\hat{x}) = \{x_0\} .$$

Note that x_0 is an unstable critical point (otherwise $L(x) = L(\hat{x})$ would consist of a single point.) Moreover it is easy to see that $x_0 \in \mathcal{X}$, and there exists an outgoing (from x_0) separatrix γ which belongs to the set $L(x)$.

Let $x_1 \in \gamma$. If $L(x)$ is not a periodic trajectory then for x_1 as well as for $\hat{x}$ we have $L(x_1) = \{x_2\}$, where $x_2 \in \mathcal{X}$, and moreover it is easy to see that the separatrix γ does not intersect any ergodic 2-face, and enters the point x_2 at the time of quitting some ergodic 3-face.
From this by the proportion 1.4.1 we get the proposition 1.4.2.
Proposition 1.4.2 is proved.

Proposition 1.4.3 *On a set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$ the two following statements hold :*

a) the semigroup $T_S^t, t \in I\!R_+$, has a finite number of periodic trajectories ;

b) each of the periodic trajectories of the semigroup $T_S^t, t \in I\!R_+$ is either stable from both sides or unstable from both sides.

Proof : Let us first note that the number of periodic trajectories intersecting ergodic 2-faces is finite. Moreover using the lemma 1.3.9 it is easy to show that any periodic trajectory intersecting an ergodic 2-face is stable from both sides.
So to prove the proposition 1.4.3 it is sufficient to consider periodic trajectories which do not intersect ergodic 2-faces.

Moreover by the proposition 1.4.1 it is sufficient to consider periodic trajectories which do not intersect 1-faces also.

Let us call two periodic trajectories similar if these trajectories intersect the same faces. It is obvious that similar periodic trajectories intersect the faces in the same order.

Note that the semigroup $T_S^t, t \in I\!R_+$, has an infinite number of periodic trajectories if and only if it has an infinite number of similar periodic trajectories.

Let γ be a periodic trajectory of the semigroup $T_S^t, t \in I\!R_+$, which does not intersect any 1-face and any ergodic 2-face. Then γ goes along ergodic 3-faces and crosses nonergodic 2-faces. Let $\Lambda(i,j)$ be a nonergodic 2-face, such that

$$\gamma \bigcap \Lambda(i,j) \neq \emptyset .$$

It easy to see that the set $\gamma \bigcap \Lambda(i,j)$ consists of a single point. Let

$$\gamma \bigcap (i,j) = \{x_1\} .$$

Let $\hat{t} = \inf \{t \in I\!R_+ : t > 0, T^t(x_1) \in \Lambda(i,j)\}$
and

$$x_2 = T^{\hat{t}}(x_1) .$$

Then $x_1 = \varphi(x_2) = \frac{x_2}{\|x_2\|}$, and moreover by (1.4.1.) we have

$$\begin{pmatrix} x_2^i \\ x_2^j \end{pmatrix} = B_1 ... B_m \begin{pmatrix} x_1^i \\ x_1^j \end{pmatrix}$$

where for any $k = 1, ...m$

$$B_k = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^p A_{ln}^s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^q$$

for some $p, q \in \{0, 1\}$, $s, l, n \in \{1, 2, 3, 4\}$

So the vector $\begin{pmatrix} x_1^i \\ x_1^j \end{pmatrix}$ is an eigenvector of the matrix $B_1 ... B_m$ with the eigenvalue

$$\lambda = e^{\mathcal{L}_\gamma} = \| x_2 \|$$

From this it easily follows that there exists an infinite number of periodic trajectories similar to γ if and only if the matrix

$$B_1 ... B_m$$

is diagonalizable and its eigenvalues are equal to each other. It easy to see that it is possible only on a set of Lebesgue measure zero in the parameter space $\mathcal{P}_d$.

Let us further note that the periodic trajectory γ is stable from both sides iff the eigenvalues of the matrix $B_1 ... B_m$ are mutually different and $\lambda = \| x_2 \|$ is a maximal

eigenvalue of the matrix $B_1 \ldots B_m$. It is unstable from both sides iff the eigenvalues of the matrix $B_1 \ldots B_m$ are mutually different and $\lambda = \| x_2 \|$ is a minimal eigenvalue of the matrix $B_1 \ldots B_m$. From this it follows that on a set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$ each of the periodic trajectories is either stable from both sides or unstable from both sides.
Proposition 1.4.3. is proved.

Let us consider unstable periodic trajectories of the semigroup $T_S^t, t \in I\!R_+$. As it had been noted any periodic trajectory intersecting an ergodic 2-face is stable, and on the set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$ there is no any periodic trajectory intersecting 1-face . Each of the periodic trajectories, which do not intersects 1-faces and ergodic 2-face, goes along three or four ergodic 3-faces and crosses by the way nonergodic 2-faces (see fig.1.4.1 and fig.1.4.2).

Proposition 1.4.4 *Let γ be a periodic trajectory which does not intersect neither 1-faces nor ergodic 2-faces, go along three ergodic 3-faces and crosses by the way nonergodic 2-faces. Then γ is stable from both sides if*

$$\mathcal{L}_\gamma > 0 ,$$

and γ is unstable from both sides if

$$\mathcal{L}_\gamma < 0 .$$

Proof : Let a periodic trajectory γ go along ergodic 3-faces $\Lambda(i,j,l), \Lambda(i,j,k)$ and $\Lambda(k,j,l)$ cross nonergodic 2-faces $\Lambda(i,j), \Lambda(j,l)$ and $\Lambda(j,k)$ (see fig. 1.4.1)
Let us consider

$$x_1 \in \Lambda(i,j) \bigcap \gamma ,$$

$$\hat{t} = \inf \{ t \in I\!R_+ : T^t(x_1) \in \Lambda(i,j), t \neq 0 \}$$

By the definition of $\mathcal{L}_\gamma$ we have

$$x_2 = e^{\mathcal{L}_\gamma} x_1 . \tag{1.4.3}$$

By (1.4.1) we have

$$\begin{pmatrix} x_2^j \\ x_2^i \end{pmatrix} = A_{ik}^j . A_{kl}^j . A_{li}^j \begin{pmatrix} x_1^j \\ x_1^i \end{pmatrix} . \tag{1.4.4}$$

It is easy to see that

$$A_{ik}^j \ A_{kl}^j \ A_{li}^j = \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \tag{1.4.5}$$

for some $a, b \in I\!R_+$.

From (1.4.3), (1.4.4) it follows that the vector $\begin{pmatrix} x_1^j \\ x_1^i \end{pmatrix}$ is an eigenvector of the matrix $A_{ik}^j\ A_{kl}^j\ A_{li}^j$ with the eigenvalue $e^{\mathcal{L}_\gamma}$. From (1.4.5) we get that the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the eigenvector of the matrix $A_{ik}^j, A_{kl}^j A_{li}^j$ with the eigenvalue 1.

So as it had been noted in the proof of the proposition 1.4.3 γ is stable if

$$e^{\mathcal{L}_\gamma} > 1 ,$$

and γ is unstable if

$$e^{\mathcal{L}_\gamma} < 1 .$$

Proposition 1.4.4 is proved

As it follows from the proposition 1.4.4. any unstable periodic trajectory γ with $\mathcal{L}_\gamma > 0$ either crosses some 1-face, or goes along four ergodic 3-faces and crosses nonergodic 2-faces (see fig. 1.4.3.).

1.5 Scattering probabilities

Let Λ be a nonergodic face having at least one outgoing face. For any face Λ_i outgoing from Λ and for any $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ let us consider the value $g_\Lambda(x, \Lambda_i)$ which is equal to the probability for the induced Markov chain $\mathfrak{L}_\Lambda$ to go to infinity from the point x along the face Λ_i. (see (1.2.10))

Proposition 1.5.1 *For any nonergodic face Λ having at least one outgoing face and for any $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$*

$$\sum_{\Lambda_i} g_\Lambda(x, \Lambda_i) = 1 , \tag{1.5.1}$$

where the summation is over all faces Λ_i outgoing from Λ.

Proof: Let us prove this proportion for the case when Λ is a nonergodic 2-face. In other cases one can prove this proportion similarly.

Let $\Lambda_1, ..., \Lambda_n$ be all faces outgoing from Λ. Let us define on the set C_Λ the following function $f : C_\Lambda \to I\!R_+$:

for any face Λ_j outgoing from Λ and for any $x \in C_\Lambda \bigcap \overline{\Lambda}_j$ we set

$$f(x) = \| x \| ,$$

and for any $x \in C_\Lambda \backslash (\bigcup_{j=1}^n \Lambda_j)$ we set

$$f(x) = \| T_\Lambda^{t_x}(x) \| - t_x ,$$

where

$$t_x = \inf \{t \in I\!R_+ : T^t_\Lambda(x) \in \bigcup_{j=1}^{n} \Lambda_j\},$$

$T^t_\Lambda(x)$ is a solution of the system (1.2.6).
It is easy to see that for any $x \in C_\Lambda$

$$-\infty < f(x) < \infty$$

and for any x,$y \in C_\Lambda$

$$| f(x) - f(y) | \leq d^* \| x - y \|$$

for some fixed $d^* > 0$.

Using the results of [1] (see [1], II, 2) one can easily show that for any $x \in C_\Lambda \bigcap \mathbf{Z}^4_+$ there exists $k(x) \in \mathbf{Z}_+$ such that

$$\sup_{x \in \mathbf{Z}^4_+ \bigcap C_\Lambda} k(x) \leq \tilde{k} < \infty ,$$

and

$$E\ f(\xi^\Lambda_x(k(x))) \geq f(x) + \epsilon \tag{1.5.2}$$

for all $x \in \mathbf{Z}^4_+ \bigcap C_\Lambda$, where $\epsilon > 0$ does not depend on x, and $\xi^\Lambda_x(t)$ is a r.w. starting from the point x and corresponding to the induced Markov chain $\mathfrak{L}_\Lambda$.

It follows from (1.5.2) and from the lemma 1.2 of [1] that there exist $\alpha > 0, \beta > 0, h > 0$ and $\delta > 0$ such that for any $t \in \mathbf{Z}_+$ and for any $x \in C_\Lambda \bigcap \mathbf{Z}^4_+$

$$P\{f(\xi^\Lambda_x(t)) > \delta t\} \geq 1 - \beta e^{-hf(x) - \alpha t} \tag{1.5.3}$$

From (1.5.3) and from the construction of the function $f : C_\Lambda \to I\!R_+$, we get for any $x \in C_\Lambda \bigcap \mathbf{Z}^4_+$ and for any $t \in \mathbf{Z}_+$ the following inequality

$$P\{\bigcup_{j=1}^{n} \{\forall \tau \geq t\ \ \xi^\Lambda_x(\tau) \in Q(\Lambda_j)\ \} \geq 1 - \beta' e^{h'\rho(x,\Lambda) - \alpha' t} \tag{1.5.4}$$

Note now that for $i \neq j$

$$P\{\forall \tau \geq t\ :\ \xi^\Lambda_x(\tau) \in Q(\Lambda_i) \bigcap Q(\lambda_j)\} \leq \tag{1.5.5}$$

$$\leq P\{\forall \tau \geq t\ :\ \xi^\Lambda_x(\tau) \in Q(\Lambda_i) \setminus \Lambda_i\},$$

and by the ergodicity of the face Λ_i

$$P\{\forall \tau \geq t\ :\ \xi^\Lambda_x(\tau) \in Q(\Lambda_i) \setminus \Lambda_i\} = 0 \tag{1.5.6}$$

From (1.5.4),(1.5.5),(1.5.6) it follows that

$$\sum_{j=1}^{n} P\{\forall\tau \geq t \ : \ \xi_x^{\Lambda}(\tau) \in Q(\Lambda_j) \ \} \geq 1 - \beta' e^{h'\rho(x,\Lambda)-\alpha' t} \tag{1.5.7}$$

From (1.5.7) by the definition of the values $g_\Lambda(x, \Lambda_j), j = 1, ...n$, we get (1.5.1). Proposition 1.5.1 is proved.

Let us consider now a nonergodic 1-face Λ having outgoing faces $\Lambda_1, ..., \Lambda_n$, and an ingoing 2-face Λ_0 $(1 \leq n \leq 2)$.

Proposition 1.5.2 *On a set of complete Lebesgue measure in the parameter space of the r.w. $\mathfrak{L}$ for any $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ and for any face Λ_j outgoing from Λ the following limit exists and does not depend on x*

$$g(\Lambda_0, \Lambda_j) = \lim_{n\to\infty} g_\Lambda(x + n e_{\Lambda,\Lambda_0}, \Lambda_j)$$

where $e_{\Lambda,\Lambda_0} \in \overline{\Lambda}_0$ is the vector perpendicular to Λ such that $\| e_{\Lambda,\Lambda_0} \| = 1$,

Proof : Let $\Lambda_0 = \Lambda(i,j), 1 \leq i < j \leq 4$. We define for any $n \in \mathbf{Z}_+$ the set

$$C_{\Lambda_0}^n = C_{\Lambda(i,j)}^n = \{x = (x^1, ..., x^4) \in I\!R_+^4 : x^i = x^j = n\}$$

Note that by definition

$$C_{\Lambda_0}^n = \{x + n e_{\Lambda,\Lambda_0} \mid x \in C_{\Lambda_0}^0\}$$

for any $n \in \mathbf{Z}_+$,

$$C_{\Lambda_0}^1 = C_{\Lambda_0},$$

and

$$C_\Lambda \bigcap \mathbf{Z}_+^4 = \bigcup_{m=0}^{\infty} (C_{\Lambda_0}^m \bigcap \mathbf{Z}_+^4)$$

Let us consider for any $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ a r.w. $\xi_x^\Lambda(t)$ starting from the point x and corresponding to the induced Markov chain $\mathfrak{L}_\Lambda$.
For any $n, m \in \mathbf{Z}_+$ and for any $x \in C_{\Lambda_0}^{n+m} \bigcap \mathbf{Z}_+^4$, $y \in C_{\Lambda_0}^m \bigcap \mathbf{Z}_+^4$ let $F_\Lambda(x, y \mid C_{\Lambda_0}^m)$ be a probability that the r.w. $\xi_x^\Lambda(t)$ at the time of first hitting of $C_{\Lambda_0}^m$ enters the point y.
By definition the sum

$$\sum_{y \in C_{\Lambda_0}^m \bigcap \mathbf{Z}_+^4} F_\Lambda(x; y \mid C_{\Lambda_0}^m)$$

is equal to the probability that the r.w. $\xi_x^\Lambda(t)$ ever hits the set $C_{\Lambda_0}^m$.
Using the construction of Lyapounov functions for r.w. in $\mathbf{Z}_+^3$ which were used in the paper [1], and using the lemma 1.2 of [1] one can easily show that for any $n, m \in \mathbf{Z}_+, n \neq 0$, and for any $x \in C_{\Lambda_0}^{n+m} \bigcap \mathbf{Z}_+^4$.

$$\sum_{y \in C^m_{\Lambda_0} \bigcap \mathbf{Z}^4_+} F_\Lambda(x; y \mid C^m_{\Lambda_0}) = 1$$

Moreover by the assumption of homogeneity (A_0) for any $n \in \mathbf{Z}_+, n > 0$, and for any $x, y \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$ the value

$$F_\Lambda(x + (n+m)e_{\Lambda,\Lambda_0}; y + me_{\Lambda,\Lambda_0} \mid C^m_{\Lambda_0})$$

does not depend on $m \in \mathbf{Z}_+$.

Let us consider the Markov chain $\mathfrak{L}_{\Lambda,\Lambda_0}$ having the state space $C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$ and one-step transition probabilities

$$p_{\Lambda,\Lambda_0}(x, y) = F_\Lambda(x + e_{\Lambda,\Lambda_0}; y \mid C^0_{\Lambda_0})$$

Note that for any $n \in \mathbf{Z}_+, n > 0$ and for any $x, y \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$

$$F_\Lambda(x + ne_{\Lambda,\Lambda_0}; y \mid C^0_{\Lambda_0}) = p^{(n)}_{\Lambda,\Lambda_0}(x, y)$$

where $p^{(n)}_{\Lambda,\Lambda_0}(x, y)$ is the n-step transition probability of the chain $\mathfrak{L}_{\Lambda,\Lambda_0}$.

Lemma 1.5.3 *The Markov chain $\mathfrak{L}_{\Lambda,\Lambda_0}$ has a single irreducible class of essential states.*

Proof : To prove this lemma it is sufficient to show that for any $x, y \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$ there exist $z \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$ and $m, n \in \mathbf{Z}_+$, such that

$$p^{(n)}_{\Lambda,\Lambda_0}(x, z) \neq 0 \quad \text{and} \quad p^{(m)}_{\Lambda,\Lambda_0}(y, z) \neq 0 \tag{1.5.8}$$

To show (1.5.8) let us note that by irreducibility of the induced Markov chain $\mathfrak{L}_{\Lambda_0}$ for any $x, y \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$ there exist $N_1, N_2, n_1, n_2 \in \mathbf{Z}_+$ and $z_1 \in C_\Lambda \bigcap \mathbf{Z}^4_+$ such that $N_1 > n_1,\ N_2 > n_2$ and

$$p^{(n_1)}_\Lambda(x + N_1 e_{\Lambda,\Lambda_0}, z_1) \neq 0$$

and

$$p^{(n_2)}_\Lambda(y + N_2 e_{\Lambda,\Lambda_0}, z_1) \neq 0\ .$$

Let $z_1 \in C^k_{\Lambda_0}$. Since $N_1 > n_1$ and $N_2 > n_2$, then for any $m_1, m_2 \in Z_+$ such that $m_1 \leq n_1, m_2 \leq n_2$, and for any $z \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$

$$p^{(m_1)}_\Lambda(x + N_1 e_{\Lambda,\Lambda_0}, z) = 0 \text{ and } p^{(m_2)}_\Lambda(y + N_2 e_{\Lambda,\Lambda_0}, z) = 0\ . \tag{1.5.9}$$

From (1.5.9) it follows in particular that $k > 0$. For any $z \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$ using (1.5.9) we get

$$p^{(N_1)}_{\Lambda_0}(x, z) = F_\Lambda(x + N_1 e_{\Lambda,\Lambda_0}, z \mid C^0_{\Lambda_0}) \geq$$

$$\geq p_{\Lambda}^{(n_1)}(x + N_1 e_{\Lambda,\Lambda_0}, z_1) F_{\Lambda}(z_1, z \mid C^0_{\Lambda_0}) = \tag{1.5.10}$$

$$= p_{\Lambda}^{(n_1)}(x + N_1 e_{\Lambda,\Lambda_0}; z_1) p^{(k)}_{\Lambda,\Lambda_0}(z_1 - k e_{\Lambda,\Lambda_0}; z)$$

and analogously

$$p^{(N_2)}_{\Lambda,\Lambda_0}(y, z) \geq p_{\Lambda}^{(n_2)}(x + N_2 e_{\Lambda,\Lambda_0}, z_1) p^{(k)}_{\Lambda,\Lambda_0}(z_1 - k e_{\Lambda,\Lambda_0}, z) \tag{1.5.11}$$

From (1.5.10) and (1.5.11) it easily follows that there exists $z \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$ for which (1.5.8) holds.
Lemma 1.5.3 is proved.

Remark : Let us note that one can not get aperiodicity of the chain $\mathfrak{L}_{\Lambda,\Lambda_0}$ from aperiodicity of the chains $\mathfrak{L}_{\Lambda}$ and $\mathfrak{L}_{\Lambda_0}$. But for the Markov chain $\mathfrak{L}_{\Lambda,\Lambda_0}$ to be aperiodic it is sufficient that for some $x \in C_{\Lambda} \bigcap \mathbf{Z}^4_+$

$$p_{\Lambda}(x + e_{\Lambda,\Lambda_0}, x) \neq 0 \, .$$

So the chain $\mathfrak{L}_{\Lambda,\Lambda_0}$ is aperiodic on a set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$.

We shall show now that the chain $\mathfrak{L}_{\Lambda,\Lambda_0}$ is ergodic on a set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$.

Lemma 1.5.4 *There exist constants $\alpha > 0$ and $c > 0$ such that for $x_0 \in C^0_{\Lambda_0} \bigcap \Lambda$ and for any $r \in I\!R_+$, $n \in \mathbf{Z}_+$, the following inequality holds*

$$\sum_{y \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+ : \|x_0 - y\| > r} p^{(n)}_{\Lambda,\Lambda_0}(x_0, y) \leq c e^{-\alpha r} \tag{1.5.12}$$

Proof : To prove lemma 1.5.4 we shall use two following lemmas.

Let us consider for any $x, y \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$, $t \in \mathbf{Z}_+$ and for any $A \subset C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$ the probability

$$F^{(t)}_{\Lambda}(x; y \mid A) = P\{\xi^{\Lambda}_x(t) = y,\ \xi^{\Lambda}_x(\tau) \notin A \quad \forall \tau = 1, ..., t-1\}$$

Lemma 1.5.5 *There exists $\mathcal{H} > 0$, such that for $x_0 \in C^0_{\Lambda_0} \bigcap \Lambda$ and for any $n, m \in \mathbf{Z}_+$*

$$\sum_{t \in \mathbf{Z}_+} F^{(t)}_{\Lambda}(x_0 + n e_{\Lambda,\Lambda_0}; x_0 + m e_{\Lambda,\Lambda_0} \mid C^0_{\Lambda_0}) \leq \mathcal{H} \tag{1.5.13}$$

Proof : Let us first note that for any $n, m \in \mathbf{Z}_+$

$$\sum_{t \in \mathbf{Z}_+} F^{(t_1)}_{\Lambda}(x_0 + n e_{\Lambda,\Lambda_0}; x_0 + m e_{\Lambda,\Lambda_0} \mid C^0_{\Lambda_0}) =$$

$$= \sum_{t_1 \in \mathbf{Z}_+} F^{(t_1)}_{\Lambda}(x_0 + n e_{\Lambda,\Lambda_0}, x + m e_{\Lambda,\Lambda_0} \mid C^0_{\Lambda_0} \bigcup \{x + m e_{\Lambda,\Lambda_0}\}) \times \tag{1.5.14}$$

$$\times \sum_{t_2 \in \mathbf{Z}_+} F_\Lambda^{(t_2)}(x_0 + me_{\Lambda,\Lambda_0}, x_0 + me_{\Lambda,\Lambda_0} \mid C^0_{\Lambda_0}) \le$$

$$\le \sum_{t \in \mathbf{Z}_+} F_\Lambda^{(t)}(x_0 + me_{\Lambda,\Lambda_0}, x_0 + me_{\Lambda,\Lambda_0} \mid C^0_{\Lambda_0})$$

Now we shall use the results of the paper [1] about r.w. in $\mathbf{Z}^3$.

From the construction of Lyapounov functions for r.w. in $\mathbf{Z}^3_+$ which were used in [1], and from the lemma 1.2 of [1] one can easily get that there exist contants $\alpha > 0$ and $c > 0$ such that for any $t \in \mathbf{Z}_+$ and for any $m \in \mathbf{Z}_+$

$$F_\Lambda^{(t)}(x_0 + me_{\Lambda,\Lambda_0}, x_0 + me_{\Lambda,\Lambda_0} \mid C^0_{\Lambda_0}) \le Ce^{-\alpha t} \tag{1.5.15}$$

From (1.5.14) and (1.5.15) it follows (1.5.13).
Lemma 1.5.5 is proved.

Lemma 1.5.6 *There exist constants $c > 0$ and $\alpha > 0$ such that for any $m \in \mathbf{Z}_+$ and for any $t \in \mathbf{Z}_+$*

$$\sum_{y \in C_\Lambda \bigcap \mathbf{Z}^4_+} F_\Lambda^{(t)}(x_0 + me_{\Lambda,\Lambda_0}; y \mid C^0_{\Lambda_0} \bigcup \Lambda_0) \le Ce^{-\alpha t} \tag{1.5.16}$$

Proof : Let us consider $x_1 = x_0 + e_{\Lambda,\Lambda_0}$, and let us consider r.w. $\xi^{\Lambda_0}_{x_1}(t)$ starting at the point x_1 and corresponding to the induced chain $\mathfrak{L}_{\Lambda_0}$. It is easy to see that

$$\sum_{y \in C_\Lambda \bigcap \mathbf{Z}^4_+} F_\Lambda^{(t)}(x_0 + me_{\Lambda,\Lambda_0}; y \mid C^0_{\Lambda_0} \bigcup \Lambda_0) \le \tag{1.5.17}$$

$$\le P\{\xi^{\Lambda_0}_{x_1}(\tau) \notin \Lambda_0 \quad \forall \tau = 1, ..., t-1\}$$

Since $\xi^{\Lambda_0}_{x_1}(t)$ is an ergodic r.w. in $\mathbf{Z}^2_+$, then using a construction of Lyapounov functions for ergodic r.w. in $\mathbf{Z}^2_+$ which was used in [1], and using the lemma 1.2 of the paper [1] one can easily show that

$$P\{\xi^{\Lambda_0}_{x_1}(\tau) \notin \Lambda_0 \quad \forall \tau = 1, ..., t-1\} \le Ce^{-\alpha t} \tag{1.5.18}$$

where the constants $\alpha > 0$ and $c > 0$ do not depend on $t \in \mathbf{Z}_+$.
From (1.5.17) and (1.5.18) we get (1.5.16)
Lemma 1.5.6 is proved.
Let us now prove the lemma 1.5.4.

Note that for any $y \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$ and for any $n \in \mathbf{Z}_+$.

$$p^{(n)}_{\Lambda,\Lambda_0}(x_0, y) = \sum_{t \in \mathbf{Z}_+} F_\Lambda^{(t)}(x_0 + ne_{\Lambda,\Lambda_0}; y \mid C^0_{\Lambda_0}) \le$$

$$\leq \sum_{m=0}^{n} \sum_{t_1 \in \mathbf{Z}_+} F_\Lambda^{(t_1)}(x + ne_{\Lambda,\Lambda_0}; x + me_{\Lambda,\Lambda_0} \mid C^0_{\Lambda_0}) \times \tag{1.5.19}$$

$$\times \sum_{t_2 \in \mathbf{Z}_+} F_\Lambda^{(t_2)}(x + me_{\Lambda,\Lambda_0}; y \mid C^0_{\Lambda_0} \bigcup \Lambda_0)$$

Note further that by the boundedness of jumps condition (assumption A_0) for any $m \in \mathbf{Z}_+$ and for any $y \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$

$$F_\Lambda^{(t)}(x + me_{\Lambda,\Lambda_0}; y \mid C^0_{\Lambda_0} \bigcup \Lambda_0) = 0 \tag{1.5.20}$$

if $t \leq \max\ \{\frac{1}{d} \parallel y - y_0 \parallel, m\}$, where $d > 0$ is a fixed constant of the assumption A_0.

From (1.5.20) using the lemma 1.5.6 we get

$$\sum_{y \in C_{\Lambda_0} \bigcup \mathbf{Z}^4_+ : \|y - x_0\| > r} \ \sum_{t \in \mathbf{Z}_+} F^t_\Lambda(x + me_{\Lambda,\Lambda_0}; y \mid C^0_{\Lambda_0} \bigcup \Lambda_0) \leq Ce^{-\alpha(r+m)} \tag{1.5.21}$$

for any $r \in I\!R_+$ and for any $m \in \mathbf{Z}_+$ where the constants $c > 0$ and $\alpha > 0$ do not depend on $r \in I\!R_+$ and on $m \in \mathbf{Z}_+$.

From (1.5.19) using (1.5.21) and lemma 1.5.4 one can easily get (1.5.12).

Lemma 1.5.4 is proved.

From irreducibility and aperiodicity of the chain $\mathfrak{L}_{\Lambda,\Lambda_0}$ on a set of complete Lebesgue measure in the parameter space $\mathcal{P}_d$ and from the lemma 1.5.4 it easily follows ergodicity of the chain $\mathfrak{L}_{\Lambda,\Lambda_0}$ on a set of complete measure in the parameter space $\mathcal{P}_d$.

Using ergodicity of the chain $\mathfrak{L}_{\Lambda,\Lambda_0}$ we shall now prove the proposition 1.5.2.

It is sufficient to prove the proposition 1.5.2 for any $x \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$.

Let us note that for any $x \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$, for any $n \in \mathbf{Z}_+$, and for any outgoing from Λ face Λ_j.

$$g_\Lambda(x + ne_{\Lambda,\Lambda_0}, \Lambda_j) = \sum_{y \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+} p^{(n)}_{\Lambda,\Lambda_0}(x, y) g_\Lambda(y, \Lambda_j)$$

From this by ergodicity of the chain $\mathfrak{L}_{\Lambda,\Lambda_0}$ we get

$$g_\Lambda(x + ne_{\Lambda,\Lambda_0}, \Lambda_j) \to \sum_{y \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+} \pi_{\Lambda,\Lambda_0}(y) g_\Lambda(y, \Lambda_j)$$

as $n \to \infty$, where $\pi_{\Lambda,\Lambda_0}(y), y \in C^0_{\Lambda_0} \bigcap \mathbf{Z}^4_+$ are the stationary probabilities of the chain $\mathfrak{L}_{\Lambda,\Lambda_0}$

Proposition 1.5.2 is proved.

Chapter 2

Auxiliary results

2.1 Movement along ergodic faces

Let Λ be an ergodic k-face ($k = 1,2,3$), and $\xi_x^\Lambda(t)$ be a r.w. starting at the point $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ and corresponding to the induced chain $\mathfrak{L}_\Lambda$.

Proposition 2.1.1 .(Kolmogorov inequality.)

Let $V_t(x,y), x,y \in C_\Lambda \bigcap \mathbf{Z}_+^4, t \in \mathbf{Z}_+$, be independent random vectors with the values in $I\!R^\nu (\nu \in \mathbf{Z}_+)$. Assume that for any $x,y \in C_\Lambda \bigcap \mathbf{Z}_+^4$ the vectors $V_t(x,y), t \in I\!R_+$ are identically distributed and

$$\| V_t(x,y) \| \leq C_0 < \infty \qquad a.s. \tag{2.1.1}$$

where $C_0 > 0$ does not depend on x,y.

Then for any $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ and for any $\delta > 0$ there exists $c = c(x,\delta) > 0$ such that for any $t \in \mathbf{Z}_+$

$$P\{ \sup_{\tau=1,\ldots,t} \| \sum_{n=0}^{\tau} V_n(\xi_x^\Lambda(n), \xi_x^\Lambda(n+1)) - EV_n(\xi_x^\Lambda(n), \xi_x^\Lambda(n+1)) \| > \delta t\} \leq \frac{c}{t} \tag{2.1.2}$$

To prove this proposition we shall need the following

Lemma 2.1.2 *For any $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ there exist $c_1 > 0$ and $\alpha_1 > 0$ such that for any $n,m \in \mathbf{Z}_+$*

$$\begin{aligned} &| E(V_n(\xi_x^\Lambda(n), \xi_x^\Lambda(n+1)) - EV_n(\xi_x^\Lambda(n), \xi_x^\Lambda(n+1)), \\ &V_m(\xi_x^\Lambda(m), \xi_x^\Lambda(m+1)) - EV_m(\xi_x^\Lambda(m), \xi_x^\Lambda(m+1))) | \leq c_1 e^{-\alpha_1 |n-m|} \end{aligned} \tag{2.1.3}$$

where $(.,.)$ is a scalar product in $I\!R^\nu$,

$$(x,y) = \sum_{j=1}^{\nu} x^j y^j, \ x,y \in I\!R^\nu .$$

Proof of the lemma 2.1.2

Let $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ be fixed, $V_n = V_n(\xi_x^\Lambda(n), \xi_x^\Lambda(n+1))$, $a_n = EV_n, n \in \mathbf{Z}_+$. For any $n, m \in \mathbf{Z}_+, n > m$, one can easily get

$$E(V_n - a_n, V_m - a_m) = \sum_{x', x'' \in C_\Lambda \bigcap \mathbf{Z}_+^4} p_\Lambda^{(m)}(x, x') p_\Lambda(x', x'') \times$$

$$\times \sum_{y', y'' \in C_\Lambda \bigcap \mathbf{Z}_+^4} (p_\Lambda^{(n-m-1)}(x'', y') - \pi_\Lambda(y')) p_\Lambda(y', y'') \times \tag{2.1.4}$$

$$\times (EV_m(x', x'') - a_m, EV_n(y', y'') - a_n)$$

where $p_\Lambda(x, y), x, y \in C_\Lambda \bigcap \mathbf{Z}_+^4$, are the one step transition probabilities of the induced chain $\mathfrak{L}_\Lambda$, and $\pi_\Lambda(y), y \in C_\Lambda \bigcap \mathbf{Z}_+^4$, are the stationary probabilities of $\mathcal{L}_\Lambda$.

From (2.1.4) by (2.1.1) it follows that

$$| E(V_n - a_n, V_m - a_m) | \leq$$

$$\leq 4C_0^2 \sum_{x', y' \in C_\Lambda \bigcap \mathbf{Z}_+^4} p_\Lambda^{(m+1)}(x, x') \times \tag{2.1.5}$$

$$\times | p_\Lambda^{(n-m-1)}(x', y') - \pi_\Lambda(y') |$$

Let us now use the results of the paper [1]. Since the chain $\mathfrak{L}_\Lambda$ satisfies all the conditions which were used in the paper [1], then the two following propositions hold

1. for any $h > 0$ there exist $c > 0$ and $h' > 0$ such that for any $x, y \in C_\Lambda \bigcap \mathbf{Z}_+^4$ and for any $t \in \mathbf{Z}_+$
$$p_\Lambda^{(t)}(x, y) \leq c e^{h\|x\| - h'\|y\|} \tag{2.1.6}$$
2. for any $h > 0$ there exist $c > 0$ and $\alpha > 0$, such that for any $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ and for any $t \in \mathbf{Z}_+$
$$\sum_{y \in C_\Lambda \bigcap \mathbf{Z}_+^4} | p_\Lambda^{(t)}(x, y) - \pi_\Lambda(y) | \leq c e^{h\|x\| - \alpha t} \tag{2.1.7}$$

From (2.1.5), (2.1.6) and (2.1.7) one can easily get (2.1.3).
Lemma 2.1.2 is proved.

Proof of the proposition 2.1.1

Let $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ be fixed, $V_n = V_n(\xi_x^\Lambda(n), \xi_x^\Lambda(n+1))$, $a_n = EV_n$ and

$$S_n = \sum_{j=0}^{n} (V_j - a_j), n \in \mathbf{Z}_+ .$$

For given $\delta > 0$ and $t \in \mathbf{Z}_+$ let us consider the following events

$$A_n = \{\| S_{n+1} \|> \delta t,\ \| S_0 \|< \delta t, ..., \| S_n \|< \delta t\}, n = 0, ..., t.$$

Let I_n be the indicator of the event A_n.
It is easy to see that

$$E \| S_t \|^2 \geq \sum_{n=0}^{t-1} E \| S_t \|^2 .I_n \geq \sum_{n=0}^{t-1} E \| S_n \|^2 I_n - \\ -2\sum_{n=0}^{t-1} \sum_{j=n+1}^{t-1} | E(S_n, V_j - a_j) | , \tag{2.1.8}$$

where

$$\| S_t \|^2 = (S_t, S_t) = \sum_{j=1}^{4} (S_t^j)^2 .$$

From (2.1.8) one can easily get

$$E \| S_t \|^2 \geq \sum_{n=0}^{t-1} E \| S_n \|^2 I_n - c'.t , \tag{2.1.9}$$

where $c' = c'(x, \delta) > 0$.

Let us note now that

$$\sum_{n=0}^{t-1} E \| S_n \|^2 I_n \geq \delta^2 t^2 P\{\max_{n=1,...,t} \| S_n \|> \epsilon t\} \tag{2.1.10}$$

From (2.1.9) and (2.1.10) we get

$$P\{\max_{n=1,...,t} \| S_n \|> \epsilon t\} \leq \frac{E \| S_t \|^2 + c' t}{\epsilon^2 t^2} \tag{2.1.11}$$

Let us now estimate the value $E \| S_t \|^2$.

$$E \| S_t \|^2 \leq \sum_{j=1}^{t} E \| V_j - a_j \|^2 + 2\sum_{j=1}^{t} \sum_{e=j+1}^{t} | E(V_j - a_j, V_e - a_e) | \tag{2.1.12}$$

From (2.1.12) using (2.1.1) and lemma 2.1.2 we get

$$E \| S_t \|^2 \leq c'' t , \tag{2.1.13}$$

where $c'' = c''(x) > 0$

From (2.1.11) and (2.1.13) we get (2.1.2)

Proposition 2.1.1 is proved.

From the proposition 2.1.1 by ergodicity of the chain $\mathfrak{L}_\Lambda$ it easily follows

Proposition 2.1.3 *Let the conditions of the proposition 2.1.1 be satisfied, and*

$$V = \sum_{x,y \in C_\Lambda \bigcap \mathbf{Z}_+^4} \pi_\Lambda(x) p_\Lambda(x,y) E V_0(x,y) .$$

Then for any $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ and for any $\delta > 0$ there exists $c = c(x,\delta) > 0$ such that for any $t \in \mathbf{Z}_+$ the following estimate holds

$$P\{ \sup_{\tau=1,\ldots,t} \| \sum_{n=0}^{\tau} V_n(\xi_x^\Lambda(n), \xi_x^\Lambda(n+1)) - V\tau \| > \delta t\} \leq \frac{c}{t} \tag{2.1.14}$$

Let us consider a r.w. $\xi_x(t)$ corresponding to the chain $\mathcal{L}$ starting at the point x. We shall use the proposition 2.1.3 to study the movement of the r.w. $\xi_x(t)$ along ergodic faces.

Proposition 2.1.4 *There exist constants $\alpha > 0$ and $\beta > 0$ and for any $\delta > 0$ there exists $c > 0$ such that for any ergodic face Λ and for any $x \in \Lambda \bigcap \mathbf{Z}_+^4$ and $t \in \mathbf{Z}_+$, for which*

$$\{T^\tau(x), 0 < \tau < t\} \subset \Lambda ,$$

the following estimate holds

$$P\{\| \xi_x(t) - T^t(x) \| > \delta t\} \leq \frac{c}{t} + \beta^{e^{-\alpha\rho(x,\partial\Lambda)}}$$

where $\partial\Lambda = \overline{\Lambda} \backslash \Lambda$.

Proof : Let, for any face Λ, $Q(\Lambda)$ be a union of all faces Λ' such that $\Lambda \subseteq \overline{\Lambda'}$, and let $\partial Q(\Lambda) = \overline{Q(\Lambda)} \backslash Q(\Lambda)$.
Let us consider for any $x \in \mathbf{Z}_+^4$ and for any $A \subseteq \mathbf{Z}_+^4$ the time $\tau_x(A)$ of first hitting of the set A by the r.w. $\xi_x(t)$.

Lemma 2.1.5 *Let Λ be an outgoing ergodic face for a face Λ'. Then there exist constants $\alpha > 0$ and $c > 0$, such that for any $x \in \Lambda \bigcap \mathbf{Z}_+^4$*

$$P\{\xi_x(\tau_x(\partial Q(\Lambda))) \in \overline{Q(\Lambda') \backslash Q(\Lambda)}\} \leq c e^{-\alpha\rho(x,\Lambda')}$$

Proof of the lemma 2.1.5

Let us consider, for any $z \in C_{\Lambda'} \bigcap \mathbf{Z}_+^4$ and for any $A \subset C_{\Lambda'} \bigcap \mathbf{Z}_+^4$, the r.w. $\xi_z^{\Lambda'}(t)$ corresponding to the induced chain $\mathfrak{L}_{\Lambda'}$ starting at the point z, and the time $\tau_z^{\Lambda'}(A)$ of first hitting of the set A by the r.w. $\xi_z^{\Lambda'}(t)$.
Let $x \in \Lambda \bigcap \mathbf{Z}_+^4$, and let $x' \in C_{\Lambda'}$ be the orthogonal projection of the point x onto $C_{\Lambda'}$. It is easy to see that for any $t \in \mathbf{Z}_+$.

$$P\{\xi_x(\tau_x(\partial Q(\Lambda))) \in \overline{Q(\Lambda') \backslash Q(\Lambda)},\ \tau_x(\partial Q(\Lambda)) = t\} \leq$$

$$\leq P\{\tau_{x'}^{\Lambda'}(C_{\Lambda'}\backslash Q(\Lambda)) = t\} \ .$$

So to prove the lemma 2.1.5 it is sufficient to show that for any $z \in C_{\Lambda'} \bigcap \mathbf{Z}_+^4$

$$P\{\tau_z^{\Lambda'}(C_{\Lambda'}\backslash Q(\Lambda)) < \infty\} \leq ce^{-\alpha\rho(z,\Lambda')} \tag{2.1.15}$$

where the constants $c > 0$ and $\alpha > 0$ do not depend on x.

To prove (2.1.15) let us defin on the set $C_{\Lambda'}$ the following function f :

(i) for any $x \in C_{\Lambda'}$ such that

$$t_x = \inf\{t \in \mathbb{R}_+ \ : T_{\Lambda'}^t(x) \in \Lambda\} < \infty \ ,$$

where $T_{\Lambda'}^t(x)$ is the solution of the system (1.2.6), we set

$$f(x) = -t_x + \parallel T_{\Lambda'}^{t_x}(x) \parallel \ ,$$

(ii) for any $x \in C_{\Lambda'}$ for which $t_x = \infty$ we set $f(x) = 0$.

Note that for any $x \in C_{\Lambda'}$, for which $f(x) \neq 0$ we have

$$\theta_1 \parallel x \parallel \leq \mid f(x) \mid \leq \theta_2 \parallel x \parallel \tag{2.1.16}$$

where the constants $\theta_1 > 0$ and $\theta_2 > 0$ do not depend on x, and there exists $d_1 > 0$ such that for any $x, y \in C_{\Lambda'}$, for which $f(x) \neq 0, f(y) \neq 0$, we have

$$\mid f(x) - f(y) \mid \leq d_1 \parallel x - y \parallel \tag{2.1.17}$$

Moreover it follows from the results of the paper [1] that there exists $\mathcal{D} > 0$ such that for any $x \in C_{\Lambda'} \bigcap \mathbf{Z}_+^4$ for which $f(x) > \mathcal{D}$ there exists $k(x) \in \mathbf{Z}_+$ such that

$$\sup_x k(x) \leq \tilde{k} < \infty \ , \tag{2.1.18}$$

and

$$Ef(\xi_x^{\Lambda'}(k(x))) \geq f(x) + \epsilon \tag{2.1.19}$$

where $\epsilon > 0$ does not depend on x.

From (2.1.16)-(2.1.19) by the lemma 1.2 of [1] it follows that for any $x \in C_{\Lambda'} \bigcap \mathbf{Z}_+^4$ for which $f(x) > 0$.

$$P\{\exists t \in \mathbf{Z}_+ : f(\xi_x^{\Lambda'}(t)) \leq 0\} \leq ce^{-\alpha\|x\|} \tag{2.1.20}$$

where the constants $c > 0$ and $\alpha > 0$ do not depend on x.

Let us note now that

$$P\{\tau_x^{\Lambda'}(C_{\Lambda'}\backslash Q(\Lambda)) < \infty\} \leq P\{\exists t \in \mathbf{Z}_+ : f(\xi_x^{\Lambda'}(t)) \leq 0\} \tag{2.1.21}$$

From (2.1.20) and (2.1.21) we get (2.1.15)
Lemma is 2.1.5 is proved.

Lemma 2.1.6 *There exist constants $h > 0, c > 0$ and $\alpha > 0$ such that for any ergodic face Λ, for any $x \in Q(\Lambda) \bigcap \mathbf{Z}_+^4$ and for any $t \in \mathbf{Z}_+$*

$$P\{\tau_x(\partial Q(\Lambda)) > t, \tau_x(\Lambda) > t\} \leq ce^{h\rho(x,\Lambda)-\alpha t}$$

This lemma follows from the results of the paper [1] (see the construction of the Lyapounov function for r.w. in $\mathbf{Z}_+^1, \mathbf{Z}_+^2$ and $\mathbf{Z}_+^3$ and lemma 1.2 of [1]).

Lemma 2.1.7 *There exist constants $h > 0, c > 0$ and $\alpha > 0$ such that for any $x \in Q(\Lambda) \bigcap \mathbf{Z}_+^4$, $r \in I\!R_+$ and for any $t \in \mathbf{Z}_+$*

$$P\{\tau_x(\partial Q(\Lambda)) > t, \rho(\xi_x(t), \Lambda) > r\} \leq ce^{h\rho(x,\Lambda)-\alpha r}$$

This lemma follows from the lemma 2.1.6 by boundedness of jumps of the r.w. $\xi_x(t)$.

Let us now prove the proposition 2.1.4.

Let

$$A_- = \bigcup_{\Lambda':\Lambda' \subset \overline{\Lambda}} Q(\Lambda') \backslash Q(\Lambda)$$

where the union is over all faces Λ' for which Λ is an ingoing face, and let

$$A_+ = \bigcup_{\Lambda':\Lambda' \subset \overline{\Lambda}} Q(\Lambda') \backslash Q(\Lambda) \, .$$

where the union is over all faces Λ' for which Λ is an outgoing face.
Note that $\overline{A}_- \bigcup \overline{A}_+ \supseteq \partial Q(\Lambda)$, and by the lemma 2.1.5 for any $x \in \Lambda \bigcap \mathbf{Z}_+^4$

$$P\{\xi_x(\tau_x(\partial Q(\Lambda))) \in \overline{A}_+\} \leq \beta e^{-\alpha\rho(x,\partial\Lambda)} \tag{2.1.22}$$

So to prove the proposition 2.1.4 it is sufficient to show that for any $\delta > 0$ there exists $c > 0$ such that for any $x \in \Lambda \bigcap \mathbf{Z}_+^4, t \in \mathbf{Z}_+$, for which $\{T^\tau(x), 0 < \tau < t\} \subset \Lambda$, the following estimate holds

$$P\{\xi_x(\tau_x(\partial Q(\Lambda))) \in \overline{A}_-, \| \xi_x(t) - T^t(x) \| > \delta t\} \leq \frac{c}{t} \tag{2.1.23}$$

To prove (2.1.23) let us note that by the proposition 2.1.3 and by the lemma 2.1.6 for any $\delta > 0$ there exists $c > 0$ such that for any $x \in \Lambda \bigcap \mathbf{Z}_+$ and $t \in \mathbf{Z}_+$ the following estimate holds

$$P\{\exists \tau \in \mathbf{Z}_+ : 0 \leq \tau \leq \min \{t, \tau_x(\partial Q(\Lambda))\}, \| \xi_x(\tau) - T^\tau(x) \| > \delta t\} \leq \frac{c}{t} \tag{2.1.24}$$

So in the case when

$$\{z \in I\!R_+^4 : \inf_{0 \leq \tau \leq t} \| z - T^t(x) \| < \delta t\} \subseteq \overline{A}_+ \bigcup Q(\Lambda) \tag{2.1.25}$$

one can easily get (2.1.23) from (2.1.24).

Let us consider the case when, for given $x \in \Lambda \bigcap \mathbf{Z}^4_+, t \in \mathbf{Z}_+$ and $\delta > 0$, (2.1.25) does not hold.
Since $\{T^{\tau}(x), 0 < \tau < t\} \subset \Lambda$, then for any $\epsilon > 0$ there exists $\epsilon_1 > 0$, such that

$$\{z \in I\!R^4_+ : \inf_{0 \leq \tau \leq (1-\epsilon)t} \| z - T^t(x) \| < \epsilon_1(1-\epsilon)t\} \subseteq \overline{A}_+ \bigcup Q(\Lambda) \tag{2.1.26}$$

and for any $\delta_1 > 0$, such that $\delta_1 < \epsilon_1$, from (2.1.24) we get

$$P\{\exists \tau \in \mathbf{Z}_+ : 0 \leq \tau \leq (1-\epsilon)t, \| \xi_x(\tau) - T^{\tau}(x) \| > \delta_1(1-\epsilon)t\} \leq \frac{c}{(1-\epsilon)t} \tag{2.1.27}$$

where $c = c(\delta_1) > 0$ does not depend on x and t. Let us note now that by the boundedness of jumps of the r.w. $\xi_x(\tau)$ for $\tau \in \mathbf{Z}_+$, such that

$$(1-\epsilon)t - 1 \leq \tau \leq (1-\epsilon)\tau \ ,$$

we have

$$\| \xi_x(t) - \xi_x(\tau) \| \leq \epsilon dt \tag{2.1.28}$$

From (2.1.27) and (2.1.28) for sufficiently small $\epsilon > 0$ and $\epsilon_1 > \delta_1 > 0$ we get (2.1.23). Proposition 2.1.4 is proved.

Let us consider for any $x \in I\!R^4_+$ and for any $A \subseteq I\!R^4_+$

$$t_x(A) = \inf \{t \in I\!R_+ : T^t(x) \in A\} \ .$$

Proposition 2.1.8 *There exist constants $\alpha > 0, \beta > 0$ and for any $\delta > 0$ there exists $c > 0$ such that for any ergodic face Λ for which the vector V_Λ has at least one negative component and for any $x \in \Lambda \bigcap \mathbf{Z}^4_+$ the following estimate holds*

$$P\{\| \xi_x(\tau_x(\partial Q(\Lambda))) - T^{t_x(\partial\Lambda)}(x) \| < \delta t_x(\partial\Lambda),$$

$$| \tau_x(\partial Q)(\Lambda)) - t_x(\partial\Lambda) | < \delta t_x(\partial\Lambda)\} \geq$$

$$\geq 1 - \frac{c}{t_x(\partial\Lambda)} - \beta e^{-\alpha\rho(x,\partial\Lambda)}$$

To prove this proposition we shall need the following lemma.

Lemma 2.1.9 *Let Λ be an ingoing face for a face Λ'. Then there exist $\alpha > 0$ and $c > 0$ such that for any $x \in \Lambda \bigcap \mathbf{Z}^4_+$ and for any $r \in I\!R_+$*

$$P\{\xi_x(\tau_x(\partial Q(\Lambda))) \in \overline{Q(\Lambda')\backslash Q(\Lambda)}, \ \rho(\Lambda, \xi_x(\tau_x(\partial Q(\Lambda)))) > r\} \leq ce^{-\alpha r}$$

For a 2-face Λ this lemma easily follows from the lemma 1.5.4. In other cases one can easily get an analog of the lemma 1.5.4 and to prove by this the lemma 2.1.9.

Using the lemmas 2.1.9, 2.1.5 and the proposition 2.1.3, similarly to the proof of the proposition 2.1.4, one can easily get the proposition 2.1.9.

2.2 Escaping from nonergodic faces and scattering

Let Λ be a nonergodic face.
As it follows from the results of [1] any nonergodic 2-face as well as any nonergodic 3-face has at least one outgoing face. For a nonergodic 1-face there exists an example when this nonergodic face has no any outgoing face.

Proposition 2.2.1 *Let Λ be a nonergodic 1-face which has no any outgoing face. Then there exist constants $\alpha > 0$, $\beta > 0$, $\delta > 0$, $\epsilon > 0$ and $\theta_\Lambda > 0$ such that for any $t \in \mathbf{Z}_+$ and for any $x \in Q(\Lambda) \bigcap \mathbf{Z}_+^4$, for which*

$$\rho(x, \Lambda) < \epsilon_\Lambda t \textit{ and } \| P_\Lambda(x) \| > \theta_\Lambda t \ ,$$

the following estimate holds

$$\begin{aligned} &P\{\tau_x(\partial Q(\Lambda)) > t, \textit{ and } \exists t' = 1..., t : \\ &\xi_x(t') \in Q(\Lambda) \backslash \Lambda(1,2,3,4), \textit{ and } \rho(\xi_x(t'), \Lambda) > \delta t\} \geq 1 - \beta e^{-\alpha t} \end{aligned} \tag{2.2.1}$$

Proof : Let Λ be a nonergodic 1-face, which has no any outgoing face. In this case all the nonzero components of the vector

$$V^{\Lambda}_{\Lambda(1,2,3,4)} = V_{\Lambda(1,2,3,4)} - P_\Lambda(V_{\Lambda(1,2,3,4)})$$

are negative (see paper [1]), where P_Λ is the orthogonal projector onto Λ.
From this by the lemma 1.2 of the paper [1] it follows that there exist constants $\alpha_1 > 0$, $\beta_1 > 0$ and $\epsilon_1 > 0$, such that for any $t \in \mathbf{Z}_+$ and for any $x \in Q(\Lambda) \bigcap \mathbf{Z}_+^4$ such that $\rho(x, \Lambda) < \epsilon_1 t$, the following estimate holds

$$\begin{aligned} &P\{\tau_x(\partial Q(\Lambda)) > t \text{ and } \forall t' \in \mathbf{Z}_+, t \leq t' \leq \tau_x(\partial Q(\Lambda)) : \\ &\xi_x(t') \in \Lambda(1,2,3,4)\} \leq \beta_1 e^{-\alpha_1 t} \end{aligned} \tag{2.2.2}$$

Let us note now that by the boundedness of jumps of the r.w. $\xi_x(t)$ for any $x \in Q(\Lambda) \bigcap \mathbf{Z}_+^4$

$$\tau_x(\partial Q(\Lambda)) \geq \frac{1}{d} \| P_\Lambda(x) \| \quad \text{a.s.} \tag{2.2.3}$$

From (2.2.2) and (2.2.3) it follows that for any $t \in \mathbf{Z}_+$ and for any $x \in Q(\Lambda) \bigcap \mathbf{Z}_+^4$, such that $\rho(x, \Lambda) < \epsilon_1 t$ and $\| P_\Lambda(x) \| > dt$ the following estimate holds:

$$\begin{aligned} &P\{\tau_x(\partial Q(\Lambda)) > t \ , \ \exists t' = t, ..., \tau_x(\partial Q(\Lambda)) : \\ &\xi_x(t') \in Q(\Lambda) \backslash \Lambda(1,2,3,4)\} \geq 1 - \beta_1 e^{-\alpha_1 t} \end{aligned} \tag{2.2.4}$$

Let us note further that from the construction of the Lyapounov function for the r.w. in $\mathbf{Z}_+^3$ (see [1]) and from the lemma 1.2 of the paper [1] it follows that there exist constants $\alpha_2 > 0,\ \beta_2 > 0,\ \epsilon_2 > 0$ and $\delta_2 > 0$ such that for any $t \in \mathbf{Z}_+$ and for any $x \in Q(\Lambda)\backslash\mathbf{Z}_+^4$ for which $\rho(x,\Lambda) < \epsilon_2 t$, we have

$$P\{\tau_x(\partial Q(\Lambda)) > t, \exists t' = t, ..., \tau_x(\sigma Q(\Lambda)) : \rho(\xi_x(t'),\Lambda) < \delta t\} \leq \beta_2 c^{-\alpha_2 t} \tag{2.2.5}$$

From (2.2.4) and (2.2.5) we get (2.2.1).

Proposition 2.2.1 is proved.

Let us consider a nonergodic face Λ having at least one outgoing face. Let $\Lambda_1, ..., \Lambda_n$ be all outgoing faces of the face Λ. For any $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$ and for any outgoing face $\Lambda_j (j = 1, ..., n)$ let us consider a probability $g_\Lambda(x,\Lambda_j)$ that the chain $\mathcal{L}_\Lambda$ starting at the point x goes to infinity along the face Λ_j (see (1.2.10)).

Proposition 2.2.2 *There exist constants $\alpha > 0,\ \beta > 0,\ \delta_\Lambda > 0,\ \sigma_\Lambda > 0,\ \epsilon_\Lambda > 0$ and $\theta_\Lambda > 0$ such that for any $t \in \mathbf{Z}_+$ and for any $x \in Q(\Lambda)\bigcap\mathbf{Z}_+^4$ such that $\rho(x,\Lambda) < \epsilon_\Lambda t$ and $\rho(x,\partial Q(\Lambda)) > \theta_\Lambda t$, the following estimate holds*

$$\begin{aligned}\sum_{j=1}^{n} \mid P\{\tau_x(\partial Q(\Lambda)) > t, \exists t' = \sigma_\Lambda t, ..., t\ :\\ \xi_x(t') \in \Lambda_j, \rho(\xi_x(t'),\partial\Lambda') > \delta_\Lambda t\} - g_\Lambda(x,\Lambda') \mid \leq \beta e^{-\alpha t}\end{aligned} \tag{2.2.6}$$

Proof : Let us prove this proportion for the case when Λ is a nonergodic 2-face. For other cases one can prove it analogously.

Let us consider the function $f : C_\Lambda \to I\!\!R_+$ which had been constructed in the proof of the proposition 1.5.1.

As it was shown in the proof of the proposition 1.5.1, there exist constants $\alpha > 0,\ \beta > 0,\ h > 0$ and $\delta > 0$ such that for any $t \in \mathbf{Z}_+$ and for any $x \in C_\Lambda \bigcap \mathbf{Z}_+^4$

$$P\{f(\xi_x^\Lambda(t)) > \delta t\} \geq 1 - \beta e^{-hf(x)-\alpha t}$$

where $\xi_x^\Lambda(t)$ is the r.w. corresponding to the chain $\mathfrak{L}_\Lambda$ starting at the point x.

From this and from the construction of the function f it follows that there exist constants $\alpha' > 0,\ \beta' > 0,\ h' > 0$ and $\delta' > 0$ such that for any $x \in C_\Lambda \bigcap \mathbf{Z}_+^4, t \in \mathbf{Z}_+$

$$\sum_{j=1}^{n} P\{\forall t' \geq t : \xi_x^\Lambda(t') \in Q(\Lambda_j)\ , \rho(\xi_x^\Lambda(t'),\ \sigma Q(\Lambda_j)) > \delta' t'\} \geq 1 - \beta' e^{h'\rho(x,\Lambda)-\alpha' t} \tag{2.2.7}$$

From this by the definition of the values $g_\Lambda(x,\Lambda_j), j = 1, ..., n$, one can easily get

$$\begin{aligned}\textstyle\sum_{j=1}^{n} \mid P\{\forall t' \geq t : \xi_x^\Lambda(t') \in Q(\Lambda_j),\\ \rho(\xi_x^\Lambda(t'),\partial Q(\Lambda_j)) > \delta' t'\} - g_\Lambda(x,\Lambda_j) \mid \leq \beta' e^{h'\rho(x,\Lambda)-\alpha' t}\end{aligned} \tag{2.2.8}$$

So in the case when the face $\Lambda(1,2,3,4)$ is an outgoing face for Λ (in this case there is no any other outgoing faces for Λ) the proposition 2.2.2 is proved by (2.2.8).

Let us consider now the case when $\Lambda(1,2,3,4)$ is an ingoing or neutral face for the face Λ.

Let $0 < \sigma < 1$, $t_1 = \sigma t$.

From the boundedness of jumps of the r.w. $\xi_x^\Lambda(t)$ it follows that for any outgoing face Λ_j $(j = 1, ..., n)$

$$\rho(\xi_x^\Lambda(t_1), \Lambda_j) \leq d\sigma t + \rho(x, \Lambda) \quad \text{a.s.}$$

From this by lemma 2.1.6 we get

$$P\{\forall t' = t_1, ..., t : \xi_x^\Lambda(t') \in Q(\Lambda_j)\backslash\Lambda_j\} \leq$$
$$\leq \beta'' \exp\{h''\rho(x, \Lambda) + h''\sigma dt - \alpha''(1-\sigma)t\} \tag{2.2.9}$$

where the constants $\alpha'' > 0$, $\beta'' > 0$, $h' > 0$ don't depend on x, t and σ.
From (2.2.8) and (2.2.9) for sufficiently small $\sigma > 0$ one can easily get

$$\sum_{j=1}^{n} | P\{\exists t' = \sigma t, ..., t : \xi_x^\Lambda(t') \in \Lambda_j, \rho(\xi_x^\Lambda(t'), \partial\Lambda_j) > \delta'\sigma t\}$$
$$-g_\Lambda(x, \Lambda_j) | \leq \beta''' \exp\{h'''\rho(x, \Lambda) - \alpha'''t\}$$

where the constants $\beta''' > 0$, $h''' > 0$ and $\alpha''' > 0$ do not depend on x and t

Proposition 2.2.2 is proved.

2.3 Law of large numbers

We shall consider here only the case when for any ergodic face Λ the vector V_Λ has at least one negative component.

Let $x \in I\!R_+^4, x \neq 0, t \in \mathbf{Z}_+, t \neq 0$. The point $x' = T^t(x)$ is said to be a break-point of the trajectory $\{T^\tau(x), t \in I\!R_+\}$ if

$$\lim_{\tau \uparrow t} V(T^\tau(x)) \neq V(T^t(x)) \, .$$

Let us consider for any $x \in I\!R_+^4, x \neq 0$, a sequence of all break-points of the trajectory $\{T^t(x), t \in I\!R_+\} : x_n = T^{t_n}(x), n = 1, 2, ...$, where $0 < t_1 < ... t_n < ....$

Let $\mathfrak{C}^2$ be the union of all nonergodic 2-faces having at least two outgoing faces. Let $\mathfrak{C}^{(1)}$ be the union of all nonergodic 1-faces Λ with $V_\Lambda = 0$ (each of these nonergodic 1-faces either has at least two outgoing faces, or has no any outgoing face).

Proposition 2.3.1 *For any $\epsilon > 0, \delta > 0$ and $N \in \mathbf{Z}_+$ there exists $c > 0$ such that*

$$P\{\| \xi_x(t) - T^t(x) \| < \delta \sum_{j=0}^{N} \| x_j \|\} \geq 1 - \sum_{j=0}^{n} \frac{c}{\| x_j \|} \tag{2.3.1}$$

for any $x_0 \in \mathbf{Z}_+^4, x_0 \neq 0$, such that $\rho(x_0, \mathfrak{C}^{(1)} \bigcup \mathfrak{C}^{(2)}) > \epsilon \| x_0 \|$, and for any $t \in \mathbf{Z}_+$, for which the trajectory $\{T^\tau(x_0), 0 < \tau < t\}$ has N break-points $x_1, ..., x_N$, such that for any $j = 1, ..., N$ $x_j \neq 0$ and $\rho(x_j, \mathfrak{C}^{(1)} \bigcup \mathfrak{C}^{(2)}) > \epsilon \| x_j \|$.

To prove this proposition we need the following lemma.

Lemma 2.3.2 *For any $\epsilon > 0$ and $\delta > 0$ there exist $\sigma > 0$ and $c > 0$, such that for any ergodic face Λ, for any $x \in \overline{\Lambda}, t \in \mathbf{Z}_+$, for which*

$$\{T^\tau(x), 0 < \tau < t\} \subset \Lambda \text{ and } \rho(x, \mathfrak{C}^{(1)} \bigcup \mathfrak{C}^{(2)}) > \epsilon t ,$$

and for any $z \in \mathbf{Z}_+^4$, such that

$$\| x - z \| < \sigma t ,$$

the following estimate holds

$$P\{\| \xi_z(t) - T^t(x) \| < \delta t\} \geq 1 - \frac{c}{t} \tag{2.3.2}$$

Proof of the lemma 2.3.2

Let Λ be an ergodic face, $x \in \overline{\Lambda}$ and $t \in \mathbf{Z}_+$ such that $\rho(x, \mathfrak{C}^{(1)} \bigcup \mathfrak{C}^{(2)}) > \epsilon t$, and

$$\{T^\tau(x), 0 < \tau < t\} \subset \Lambda . \tag{2.3.3}$$

Let us consider two following sets

$$A_+ = \bigcup_{\Lambda' : \Lambda' \subset \overline{\Lambda}} Q(\Lambda) \backslash Q(\Lambda)$$

where the union is over all faces Λ' for which the face Λ is an outgoing face, and

$$A_- = \bigcup_{\Lambda' : \Lambda' \subset \overline{\Lambda}} Q(\Lambda) \backslash Q(\Lambda)$$

where the union is over all faces Λ', for which the face Λ is an ingoing face. Then we have $\overline{A}_+ \bigcup \overline{A}_- \supseteq \partial Q(\Lambda)$.
From (2.3.3) it easily follows that

$$\rho(x, A_-) \geq \delta_0 t$$

where the constant $\delta_0 > 0$ does not depend on x and on t.

Let us consider $t_1 = \epsilon_1 t$, where $0 < \epsilon_1 < min\{\frac{\epsilon}{2d}, \frac{\delta_0}{2d}\}$.
Then by boundedness of jumps of the r.w. $\mathfrak{L}$ for any $z \in \mathbf{Z}_+^4$, such that $\| z - x \| < \sigma_1 t$, where $0 < \sigma_1 < min\{\frac{\epsilon}{2}; \frac{\delta_0}{2}\}$, we have

$$\inf_{\tau = 0, \ldots, t_1} \rho(\xi_z(\tau), \mathfrak{C}^{(1)} \bigcup \mathfrak{C}^{(2)} \bigcup A_-) > t \min \{\frac{\epsilon}{2}; \frac{\delta_0}{2}\} \quad \text{a.s.} \tag{2.3.4}$$

and

$$\sup_{\tau = 0, \ldots, t_1} \| \xi_z(\tau) - x \| \leq (\sigma_1 + \epsilon_1 d) t \quad \text{a.s.} \tag{2.3.5}$$

Moreover, using the proposition 2.2.2, one can easily show that there exist constants $c > 0, \delta_1 > 0$ and $\sigma_2 > 0, \sigma_2 < 1$, such that for any $z \in \mathbf{Z}_+^4$, for which $\| x - z \| < \sigma_2 t$, the following estimate holds

$$P\{\exists \tau = 0, ..., t_1 : \xi_z(\tau) \in \Lambda, \text{ and } \rho(\xi_z(\tau), A_+) > \delta_1 t_1\} \geq 1 - \frac{c}{t_1} \tag{2.3.6}$$

So by (2.3.4), (2.3.5) and (2.3.6) for any $z \in \mathbf{Z}_+^4$, for which $\| x - z \| < \sigma_2 t$, with a probability $1 - \frac{c}{t}$ there exists $\tau \in \mathbf{Z}_+$ such that

$$0 \leq \tau \leq \epsilon_1 t, \quad \xi_z(\tau) \in \Lambda,$$

$$\rho(\xi_z(\tau), \partial\Lambda) > t \min\{\frac{\epsilon}{2}, \frac{\delta_0}{2}, \delta_1 \epsilon_1\}$$

and

$$\| \xi_z(\tau) - x \| \leq (\sigma_1 + \epsilon_1 d)t .$$

From this for sufficiently small $\epsilon_1 > 0$ and $\sigma_1 > 0$ using the proposition 2.1.4 one can easily get (2.3.2).

Lemma 2.3.2 is proved.

From the lemma 2.3.2 one can easily get the proposition 2.3.1 using induction with respect to the number of break-points.

Chapter 3

Almost deterministic random walks

3.1 Transience for stable cycles

Let there exist a stable periodic trajectory γ of the one-parameter semigroup $T_S^t, t \in \mathbb{R}_+$, such that

$$\mathcal{L}_\gamma > 0 \tag{3.1.1}$$

Let us show that in this case the chain $\mathfrak{L}$ is transient.

Note that any periodic trajectory goes through some ergodic 2 or 3-faces crossing some nonergodic 2 or 1-faces.

Let $\varphi_0 \in \gamma$ belongs to some nonergodic 2 or 1-face,and let

$$t_\gamma^{(S)} = \inf \{t \in \mathbb{R}_+ : t \neq 0, \quad T_S^t(\varphi_0) = \varphi_0\} .$$

Consider the sequence of all ergodic 2 or 3-faces $\Lambda_0, ..., \Lambda_N$ which are intersected in turn by the trajectory $\{T_S^t(\varphi_0), 0 < t < t_\gamma^{(S)}\}$. For any $i = 0, ..., N$ let φ_{i+1} be the point where the trajectory $\{T_S^t(\varphi_0), 0 < t < t_\gamma^{(S)}\}$ hits the face Λ_i ($\varphi_{i+1} \in \overline{\Lambda}_i \bigcap \overline{\Lambda}_{i+1}$). And let $\tilde{\Lambda}_i$ be the face containing the point $\varphi_i, i = 0, ..., N$. By definition $\varphi_0 = \varphi_{N+1}, \Lambda_{N+1} = \Lambda_0, \tilde{\Lambda}_{N+1} = \tilde{\Lambda}_0$. For any $i = 0, ..., N$ there are three possibilities : either Λ_i, Λ_{i+1} are ergodic 3-faces and $\tilde{\Lambda}_{i+1}$ is a nonergodic 2-face, or Λ_i is an ergodic 3-face, Λ_{i+1} is an ergodic 2-face and $\Lambda_{i+1} = \tilde{\Lambda}_{i+1}$, or Λ_i is an ergodic 2-face and $\tilde{\Lambda}_{i+1}$ is a nonergodic 1-face. By the assumption A_9 the case when for some $i = 0, ..., N$ Λ_i is an ergodic 3-face and $\tilde{\Lambda}_{i+1}$ is a nonergodic 1-face we do not consider.

Let us consider for any $i = 0, ..., N$ and for any $\varphi \in Q(\tilde{\Lambda}_i) \cap S_+^4$

$$t(\varphi) = \inf \{t \in \mathbb{R}_+ : T_S^t(\varphi) \in \partial Q(\tilde{\Lambda}_i)\},$$

where $Q(\Lambda)$ is the union of all faces Λ' such that $\Lambda \subseteq \overline{\Lambda'}$,and $\partial Q(\Lambda) = \overline{Q(\Lambda)} \setminus Q(\Lambda)$.

Note that the semigroup $T_S^t, t \in \mathbb{R}_+$, has a finite number of periodic trajectories (see assumption A_6). Then from the stability of the periodic trajectory γ it easily follows that for any $\epsilon > 0$ there exist $\epsilon_0 > 0, ..., \epsilon_N > 0$ and $\delta > 0$, such that

$$\max_{i=0,...,N} \epsilon_i < \epsilon, \qquad \min_{i=0,...,N} \epsilon_i > \delta$$

and for any $i = 0, ..., N, \varphi \in Q(\Lambda_i) \cap S^4_+$, for which $\| \varphi - \varphi_i \| < \epsilon_i$,

$$T^{t(\varphi)}(\varphi) \in \tilde{\Lambda}_{i+1} \text{ and } \| T_S^{t(\varphi)}(\varphi) - \varphi_{i+1} \| < \epsilon_{i+1} - \delta$$

From this one can easily get

Proposition 3.1.1 *For any $\epsilon > 0$ there exist $\epsilon_0 > 0, ..., \epsilon_N > 0$ and $\delta > 0$, such that*

$$\max_{i=0,...,N} \epsilon_i < \epsilon \textit{ and } \qquad \min_{i=0,...,N} \epsilon_i > \delta \ ,$$

and for any $i = 0, ..., N, x \in I\!R^4_+$ for which $\| \varphi(x) - \varphi_i \| < \epsilon_i$,

$$T^{t_x(\partial Q(\tilde{\Lambda}_i))} \in \tilde{\Lambda}_{i+1}.$$

Moreover the following inclusion holds:

$$\{z \in I\!R^4_+ : \| T^{t_x(\partial Q(\tilde{\Lambda}_i))}(x) - z \| < \delta \| x \|\} \subseteq \{z \in I\!R^4_+ : \| \varphi(z) - \varphi_{i+1} \| < \epsilon_{i+1}\}$$

where $\varphi(x) = \frac{x}{\|x\|}$, $t_x(\partial Q(\tilde{\Lambda}_i)) = inf \, \{t \in I\!R_+ : T^t(x) \in \overline{Q(\tilde{\Lambda}_i)} \backslash Q(\tilde{\Lambda}_i)\}$.

Let us consider $\epsilon > 0$ such that for any $0 \le i < j \le N$

$$\{z \in I\!R^4_+ : z \neq 0, \| \varphi(z) - \varphi_i \| < \epsilon\} \cap \{z \in I\!R^4_+ : z \neq 0, \| \varphi(z) - \varphi_j \| < \epsilon\} = \emptyset \ .$$

Let $\epsilon_0 > 0, ..., \epsilon_n > 0, \delta > 0$ be the constants , satisfying the proposition 3.1.1 for this $\epsilon > 0$.

Let $\sigma > 0, \quad \sigma < \min_{i=0,...N} \epsilon_i$.
For any $i = 0, ..., N$ let us consider the set

$$Y_i(\epsilon_i, \sigma) = \{z \in I\!R^4_+ : z \neq 0, \rho(z, \tilde{\Lambda}_i) < \sigma \| z \|, \text{ and } \| \varphi(z) - \varphi_i \| < \epsilon_i\}.$$

For any $x \in Y_i(\epsilon_i, \sigma)$ consider a sequence $t_n(x) \in I\!R_+$, where

$$t_0(x) = 0 \ ,$$

$$t_1(x) = t_x(\partial Q(\tilde{\Lambda}_i))$$

and for any $n \in \mathbf{Z}_+$

$$t_{n+1}(x) = t_n(x) + t_1(T^{t_n(x)}(x)) \ .$$

By the proposition 3.1.1 for any $x \in \bigcup_{i=0}^{n} Y_i(\epsilon_i, \sigma)$ and for any $n \in \mathbf{Z}_+$ $t_n(x)$ is defined.
Then for any $j = 0, ..., N$ we have

$$\| T^{t_N(\varphi_j)}(\varphi_j) \| = e^{\mathcal{L}_\gamma} > 1. \tag{3.1.2}$$

Moreover since for any $t \in I\!R_+$ the map $T^t : I\!R^4_+ \to I\!R^4_+$ is continuous in any point of the set $\{z \in I\!R^4_+ : \varphi(z) \in \gamma\}$, then for sufficiently small $\epsilon_1 > 0, ..., \epsilon_N > 0$ there exists $\mathcal{H}^* > 1$ such that

$$\frac{1}{\| x \|} \| T^{t_N(x)}(x) \| \geq \mathcal{H}^* . \tag{3.1.3}$$

for any $x \in \bigcup_{j=0}^{N} Y_j(\epsilon_j, \sigma)$.

Let $\epsilon_1 > 0, ..., \epsilon_N > 0$ be sufficiently small, such that for any $x \in \bigcup_{j=0}^{N} Y_j(\epsilon_j, \sigma)$ (3.1.3) holds.
Let us consider for any $i = 0, ..., N$ and for any $x \in Y_i(\epsilon_i, \sigma) \bigcap \mathbf{Z}^4_+$ the r.w. $\xi_x(t)$ corresponding to the chain $\mathfrak{L}$ starting at the point x, and let us consider the time $\tau_1(x)$ when the r.w. $\xi_x(t)$ hits the set $(Y_{i+1}(\epsilon_{i+1}, \sigma) \bigcap \partial Q(\tilde{\Lambda}_i)) \bigcup \{0\}$ at the first time.

Note that if, for some $x \in \bigcup_{j=0}^{N} Y_i(\epsilon_j, \sigma) \bigcap \mathbf{Z}^4_+$,

$$P\{\tau_1(x) < \infty\} < 1$$

then the Markov chain $\mathfrak{L}$ is transient by definition. So to prove a transience of the chain $\mathfrak{L}$ it is sufficient to consider the case when for any $j = 0, ..., N$ and for any $x \in Y_i(\epsilon_i, \sigma) \bigcap \mathbf{Z}^4_+$

$$\tau_1(x) < \infty \quad \text{a.s.}$$

Let us consider for any $x \in (\bigcup_{i=0}^{N} Y_i(\epsilon_i, \sigma) \bigcup \{0\}) \bigcap \mathbf{Z}^4_+$ a sequence of stopping times of the r.w. $\xi_x(t)$ $\tau_n(x), n \in \mathbf{Z}_+$, where

$$\tau_0(x) = 0 \ , \ \tau_1(0) = 1 \ ,$$

and for any $n \in \mathbf{Z}_+$

$$\tau_{n+1}(x) = \tau_n(x) + \tau_1(\xi_x(\tau_n(x))) \ .$$

By definition a random sequence

$$\eta_x(n) = \xi_x(\tau_{Nn}(x)), \ n \in \mathbf{Z}_+ \ ,$$

is a Markov chain.

It is easy to see that for any $x \neq \underline{0}$ the probability of ever hitting $\underline{0}$ are equal for $\xi_x(t)$ and for $\eta_x(n)$. So it is sufficient to prove transience of $\eta_x(n)$.

To prove transience of $\eta_x(n), n \in \mathbf{Z}_+$, we shall use Foster's criterion [4] recalled thereafter:

Proposition 3.1.2 *If for a Markov chain with state space X and transition probabilities $p_{ij}, i, j \in X$, there exists a positive function f defined on the state space X and a set $A \subset X$, such that*

$$\textstyle\sum_{j \in X} p_{ij}\ f_j \leq f_i \qquad \forall\, i \in X \setminus A$$

$$\textit{and } \inf_{i \in A} f_i > \sup_{j \in X \setminus A} f_j\ ,$$

then the Markov chain is transient.

We define f on the state space of $\eta_x(n)$ by setting

$$f(x) = \frac{1}{log \ \| x \|} \text{ for } \| x \| > 3\ , \text{ and } f(x) = 1 \text{ otherwise.}$$

So to prove transience of $\eta_x(n)$ it is sufficient to prove the following proposition.

Proposition 3.1.3 *There exist $\mathcal{D} > 0$ and $\sigma > 0$ such that for any $j = 0, ..., N$ and for any $x \in Y_j(\epsilon_j, \sigma) \bigcap \mathbf{Z}_+^4$, such that $\| x \| > \mathcal{D}$, the following inequality holds*

$$E\ f(\eta_x(1)) \leq f(x) \tag{3.1.4}$$

Proof.

Let us consider for any $j = 0, ..., N$ and for any $x \in Y_j(\epsilon_j, \sigma) \bigcap \mathbf{Z}_+^4$ the time $\tau(x) = \tau_x(\partial Q(\tilde{\Lambda}_j))$ when the r.w. $\xi_x(t)$ hits the set $\partial Q(\tilde{\Lambda}_j) = \overline{Q(\tilde{\Lambda}_j)} \backslash Q(\tilde{\Lambda}_j)$ at the first time.

Lemma 3.1.4 *For any $\delta > 0$ there exist $\sigma > 0$ and $c > 0$ such that for any $j = 0, ..., N$ and for any $x \in Y_j(\epsilon_j, \sigma) \bigcap \mathbf{Z}_+^4$*

$$\begin{aligned} &P\{\| \xi_x(\tau(x)) - T^{t_1(x)}(x) \| < \delta \| x \|\ , \\ &\rho(\xi_x(\tau(x)), \tilde{\Lambda}_{j+1}) < \sigma \| \xi_x(\tau(x)) \| \} \geq 1 - \frac{c}{\| x \|} \end{aligned} \tag{3.1.5}$$

Proof of the lemma 3.1.4

Let $x \in Y_j(\epsilon_j, \sigma) \bigcap \mathbf{Z}_+^4$.

Since the constants $\epsilon_0 > 0, ..., \epsilon_N > 0$ have been chosen so that for any $i = 0, ..., N$

$$\{z \in I\!R_+ : z \neq 0, \| \varphi(z) - \varphi_i \| < 2\epsilon_i\} \subset Q(\tilde{\Lambda}_i)\ ,$$

then there exists $\theta_0 > 0$ such that

$$\tau(x) = \tau_x(\partial Q(\tilde{\Lambda}_i)) > \theta_0 \parallel x \parallel \quad \text{a.s.}$$

Let $0 < \theta < \theta_0$. Then from the proposition 2.2.2 it easily follows that there exist constants $c_1 > 0, \delta_1 > 0$ and $\sigma_1 > 0, \sigma_1 < 1$, such that for any $j = 0, ..., N$ and for any $x \in Y_j(\epsilon_j, \sigma) \bigcap \mathbf{Z}_+^4$, for which $\rho(x, \tilde{\Lambda}_j) < \sigma_1 \parallel x \parallel$,

$$\begin{aligned} &P\{\exists \tau = 0, ..., \theta \parallel x \parallel : \xi_x(\tau) \in \Lambda_i \text{ and} \\ &\rho(\xi_x(\tau), \tilde{\Lambda}_i) > \delta_1 \parallel x \parallel\} \geq 1 - \tfrac{c_1}{\|x\|} v \end{aligned} \tag{3.1.6}$$

Note also that by boundedness of jumps for any $x \in Y_i(\epsilon_i, \sigma) \bigcap \mathbf{Z}_+^4$

$$\sup_{\tau=0,...,\theta\|x\|} \parallel \xi_x(\tau) - x \parallel \leq \theta d \parallel x \parallel, \quad a.s. \tag{3.1.7}$$

where $d > 0$ is the fixed constant from the assumption A_0. Moreover for any $z \in I\!R_+^4$, such that $\parallel z - x \parallel < \theta d \parallel x \parallel$,

$$\parallel T^{t_1(z)}(z) - T^{t_1(x)}(x) \parallel < b\theta d \parallel x \parallel \tag{3.1.8}$$

where the constant $b > 0$ does not depend on x and z.

For sufficiently small $\theta > 0$ from (3.1.6), (3.1.7) and (3.1.8) using the proposition 2.1.8 one can easily get (3.1.5).

Lemma 3.1.4 is proved.

Let us consider $\delta > 0$ for which the proposition 3.1.1 holds. Then, by lemma 3.1.4 for any $0 < \delta' < \delta$ there exist $\sigma > 0$ and $c > 0$, such that for any $x \in \bigcup_{j=0}^{N} Y_i(\epsilon_i, \sigma) \bigcap \mathbf{Z}_+^4$ the following estimate holds

$$P\{\parallel \xi_x(\tau_1(x)) - T^{t_1(x)}(x) \parallel < \delta' \parallel x \parallel\} \geq 1 - \frac{c}{\parallel x \parallel} \tag{3.1.9}$$

From (3.1.9) by induction with respect to $k = 1, ..., N$ one can easily get that for any $\delta' > 0, \delta' < \delta$, there exist $\sigma > 0$ and $c > 0$ such for any $x \in \bigcup_{j=0}^{N} Y_j(\epsilon_j, \sigma) \bigcap \mathbf{Z}_+^4$

$$P\{\parallel \xi_x(\tau_N(x)) - T^{t_N(x)}(x) \parallel < \delta' \parallel x \parallel\} \geq 1 - \frac{c}{\parallel x \parallel} \tag{3.1.10}$$

Let $\mathcal{H} = \mathcal{H}^* - \delta' > 1$. Then from (3.1.10) and (3.1.3) we get

$$P\{\parallel \xi_x(\tau_N(x)) \parallel > \mathcal{H} \parallel x \parallel\} \geq 1 - \frac{c}{\parallel x \parallel} \tag{3.1.11}$$

for any $x \in \bigcup_{j=0}^{N} Y_j(\epsilon_j, \sigma) \bigcap \mathbf{Z}_+^4$.

From (3.1.11) it follows that for any $x \in \bigcup_{j=0}^{N} Y_j(\epsilon_j, \sigma) \bigcap \mathbf{Z}_+^4$, such that $\parallel x \parallel > 3$,

$$f(\eta_x(1)) = Ef(\xi_x(\tau_N(x))) \leq \frac{1}{\log\mathcal{H} \parallel x \parallel} + \frac{c}{\parallel x \parallel}$$
$$\leq \frac{1}{\log \parallel x \parallel} - \frac{\log\mathcal{H}}{(\log \parallel x \parallel)^2} + o(\frac{1}{(\log \parallel x \parallel)^2}) \tag{3.1.12}$$

By (3.1.12) we get (3.1.4) for sufficiently large $\parallel x \parallel$.

Proposition 3.1.3. is proved.

From the proposition 3.1.3 transience of the chain $\mathfrak{L}$ follows.

So the theorem 1.2.4 is proved.

3.2 Ergodicity for weakly acyclic case

Let the following conditions be satisfied :

a) the associated graph G has no cycles ;

b) for any periodic trajectory γ of the semigroup $T_S^t, t \in I\!R_+$,

$$\mathcal{L}_\gamma < 0 ;$$

c) for any stable isolated critical point $x_0 \in S_+^4$

$$(V(x_0), x_0) < 0 .$$

As it was noted in §1.2 the condition (c) is satisfied if and only if for any ergodic face Λ the vector V_Λ has at least one negative component.

We shall show that in this case the r.w $\mathfrak{L}$ is ergodic.

To prove the ergodicity we shall use the following criterion [5].

Proposition 3.2.1 *A Markov chain ζ_n with the state space X and transition probabilities $p_{ij}, i, j \in X$, is ergodic iff there exists a positive function f defined on the state space X, a finite set $A \subset X$, a sequence of stopping times $\{\tau_n\}_{n \in \mathbf{Z}_+}$ of the chain ζ_n, and a constant $\epsilon > 0$ such that for any $n \in \mathbf{Z}_+$*

i) $\tau_{n+1} - \tau_n \geq 1$ a.s.

ii) $E(f(\zeta_{\tau_{n+1}}) \mid \zeta_{\tau_n} = x) \leq f(x) - \epsilon E(\tau_{n+1} - \tau_n \mid \zeta_{\tau_n} = x)$ for any $x \in X \backslash A$.

We shall consider the following function f :

$$f(x) = \parallel x \parallel \text{ for any } x \in I\!R_+^4 .$$

To define stopping times let us consider for any $x \in \mathbf{Z}_+^4$ the r.w. $\xi_x(t)$ corresponding to the chain $\mathcal{L}$ starting at the point x, and for any $\epsilon > 0, \epsilon > 1$, let us consider the time τ_x^ϵ when the r.w. $\xi_x(t)$ hits the set $\{y \in \mathbf{Z}_+^4 : \parallel y \parallel \leq \epsilon \parallel x \parallel\}$ at the first time.

Proposition 3.2.2 *For any $\epsilon > 0, \epsilon < 1$, there exist $c > 0$ and $\theta > 0$ such that for any $x \in \mathbf{Z}^4_+, x \neq 0$,*

$$P\{\tau^\epsilon_x > \theta \parallel x \parallel\} \leq \frac{c}{\parallel x \parallel}$$

Proof. To prove this proposition we shall need the following

Lemma 3.2.3 *Let Λ be an ergodic 1-face.*

Then there exist $\delta_\Lambda > 0, \theta_\Lambda > 0$ and for any $\epsilon > 0$ there exists $c > 0$ such that for any $x \in \mathbf{Z}^4_+, z \neq 0$, where

$$\rho(z, \Lambda) < \delta_\Lambda \parallel z \parallel ,$$

the following estimate holds

$$P\{\tau^\epsilon_z > \theta_\Lambda \parallel z \parallel\} \leq \frac{c}{\parallel z \parallel} .$$

This lemma easily follows from the lemma 2.3.2

For any ergodic 1-face Λ let us fix the constants $\theta_\Lambda > 0$ and $\delta_\Lambda > 0$ from the lemma 3.2.2 and let us consider the sets

$$O_\Lambda = \{z \in I\!R^4_+ : z \neq 0,\ \rho(z, \Lambda) < \delta_\Lambda \parallel z \parallel\},$$

$$O^*_\Lambda = \{z \in I\!R^4_+ : z \neq 0,\ \rho(z, \Lambda) < \frac{\delta_\Lambda}{2} \parallel z \parallel\}, \text{ and}$$

$$O_0 = \bigcup_\Lambda Q_\Lambda \ , \ O^*_0 = \bigcup_\Lambda O^*_\Lambda \ ,$$

where the union is over all ergodic 1-faces.

Let us now consider the set $\mathcal{X}$ of vertices of the associated graph $\mathcal{G}$. For any $x \in \mathcal{X}$ let $\Lambda^{(1)}_x$ be the 1-face containing the point x.

Let us consider for any $x \in \mathcal{X}$ the set $W_-(x)$ of all separatrices outgoing from x, the set $W_+(x)$ of all separatrices ingoing to x and

$$W_+ = \bigcup_{x \in \mathcal{X}} W_+(x) \ , \ W_- = \bigcup_{x \in \mathcal{X}} W_-(x) \ .$$

Since the associated graph $\mathcal{G}$ has no cycles, then there exists $x \in \mathcal{X}$ for which

$$W_+(x) \subseteq W_+ \backslash W_- \ .$$

Let us denote the set of all these $x \in \mathcal{X}$ by $\mathcal{X}_0$.

Lemma 3.2.4 *Let $x \in \mathcal{X}_0$. Then there exist $\theta_x > 0$ and $\delta_x > 0$ and for any $\epsilon > 0$ there exists $c_x > 0$ such that for any $z \in \mathbf{Z}^4_+, z \neq 0$, for which $\parallel \varphi(z) - x \parallel < \delta_x$, the following estimate holds*

$$P\{\tau^{\epsilon}(z) > \theta_x \parallel z \parallel\} \leq \frac{c_x}{\parallel z \parallel} \tag{3.2.1}$$

Proof of the lemma 3.2.4

Let $\epsilon > 0$ be fixed, $\epsilon < 1$.

Since for any $z \in \mathbf{Z}_+^4$ and for any $\epsilon' \geq \epsilon > 0$

$$\tau^{\epsilon}(z) \geq \tau^{\epsilon'}(x) \qquad \text{a.s.},$$

then it is sufficient to prove lemma 3.2.2 for sufficiently small $\epsilon > 0$. So we shall assume $\epsilon > 0$ to be sufficiently small.

Let us consider for any $z \in I\!R_+^4, z \neq 0$, the time

$$t_z = \inf \{t \in I\!R_+ : \text{ either } \parallel T^t(z) \parallel < \frac{\epsilon}{2} \parallel z \parallel \text{ , or } T^t(z) \in O_0^*\} \ .$$

It is easy to see that there exist $\delta_1 > 0$ and $\theta_1 > 0$ such that for any face Λ outgoing from $\Lambda_x^{(1)}$ and for any $z \in \Lambda$, for which

$$\parallel \varphi(z) - x \parallel < \delta$$

the inequality

$$t_z < \theta_1 \parallel z \parallel$$

holds and the constants $\theta_1 > 0$ and $\delta_1 > 0$ do not depend on $\epsilon > 0$. It is also clear that for sufficiently small $\delta_1 > 0$ there exist $\sigma > 0$ and $N \in \mathbf{Z}_+$ such that for any face Λ outgoing from $\Lambda_x^{(1)}$ and for any $z \in \Lambda$, for which

$$\parallel \varphi(z) - x \parallel < \delta_1,$$

the trajectory $\{T^t(x), 0 < t < t_z\}$ has at most N break-points:

$$z_1 = T^{t_1}(z), ..., z_k = T^{t_k}(z),$$

$$0 \leq k \leq N,$$

$$0 < t_1 < ... < t_k < t_z \ .$$

Moreover for any break-point $z_j, j = 1, ..., k$,

$$\rho(\varphi(z_j), \ \mathfrak{C}^{(1)} \bigcup \mathfrak{C}^2) > \sigma,$$

where $\mathfrak{C}^{(1)}$ is the union of all 1-faces Λ' for which $V_{\Lambda'} = 0$, and $\mathfrak{C}^{(2)}$ is the union of all 2-face having at least two outgoing faces.

From this using the proposition 2.3.1 one can easily show that for sufficiently small $\delta_1 > 0$ and for any $\sigma_1 > 0, \sigma_1 < \delta_1$, there exists $c_1 > 0$ such that for any face Λ outgoing from $\Lambda_x^{(1)}$ and for any $z \in \Lambda \bigcap \mathbf{Z}_+^4$, for which

$$\sigma_1 < \| \varphi(z) - x \| < \delta_1 ,$$

the following estimate holds

$$P\{\tau_z^\epsilon > \theta_1 \| z \| \text{ and } \forall t = 1, ..., \theta_1 \| z \| : \ \xi_z(t) \notin O_0\} \le \frac{c_1}{\| z \|} . \tag{3.2.2}$$

Let us note now that from the proposition 2.2.2 it easily follows that for any $\delta_1 > 0$ there exist $\delta_2 > 0, c_2 > 0, \theta_2 > 0$ and $\sigma_1 > 0, \sigma_1 < \delta_1$ such that for any $z \in \mathbf{Z}_+^4, z \neq 0$, for which $\| \varphi(z) - x \| < \delta_2$.

$$P\{\exists\, t = 1, ..., \theta_2 \| z \|: \xi_z(t) \in \bigcup_\Lambda \Lambda \ , \ \sigma_1 < \| \varphi(\xi_z(t)) - x \| < \delta_1\} \ge 1 - \frac{c_2}{\| z \|} . \tag{3.2.3}$$

where the union is over all faces Λ outgoing from $\Lambda_x^{(1)}$.

Note also that for any $t = 1, ..., \theta_2 \| z \|$ by the boundedness of jumps

$$\| \xi_z(t) \| \le (1 + \theta_2 d) \| z \| \tag{3.2.4}$$

From (3.2.2), (3.2.3) and (3.2.4) it follows that for any $z \in \mathbf{Z}_+^4, z \neq 0$, for which $\| \varphi(z) - x \| < \delta_2$, the following estimatin holds

$$\begin{aligned} &P\{\tau_z^\epsilon > \theta_1(1 + \theta_2 d) \| z \| \text{ and} \\ &\forall t = 1, ..., \theta_1(1 + \theta_2 d) \| z \| : \ \xi_z(t) \notin O_0\} \le \frac{c_3}{\| z \|} \end{aligned} \tag{3.2.5}$$

where the constant $c_3 > 0$ does not depend on z.

From (3.2.5) using the lemma 3.2.3 one can easily get (3.2.1).

Lemma 3.2.4 is proved.

For any $x \in \mathcal{X}_0$ let us fix the constants $\theta_x > 0$ and $\delta_x > 0$ from the lemma 3.2.4. And let us consider the sets

$$O_x = \{z \in \mathbb{R}_+^4 : z \neq 0, \| \varphi(z) - x \| < \delta_x\},$$

$$O_x^* = \{z \in \mathbb{R}_+^4 : z \neq 0, \| \varphi(z) - x \| < \frac{\delta_x}{2}\},$$

and

$$O_1 = \bigcup_{x \in \mathcal{X}_0} O_1 \ , \ O_1^* = \bigcup_{x \in \mathcal{X}_0} O_x^* .$$

Lemma 3.2.5 *For any $x \in \mathcal{X} \backslash \mathcal{X}_0$ there exist $\theta_x > 0$ and $\delta_x > 0$ and for any $\epsilon > 0$ there exist $c_x = c_x(\epsilon) > 0$ such that for any $z \in \mathbf{Z}_+^4, z \neq 0$, for which $\| \varphi(z) - x \| < \delta_x$,*

$$P\{\tau_z^\epsilon > \theta_x \| z \|\} \leq \frac{c_x}{\| z \|} \tag{3.2.6}$$

Proof of the lemma 3.2.5.

Let us consider for any $z \in I\!R_+^4, z \neq 0$,

$$t_z = \inf \{t \in I\!R_+ : \text{ either } \| T^t(z) \| < \frac{\epsilon}{2} \| z \| \text{ , or } T^t(z) \in O_0^* \bigcup O_1^*\}.$$

Note that for any $x \in \mathcal{X} \setminus \mathcal{X}_0$ there exist $\delta_x > 0$ and $\theta_x > 0$ such that for any face Λ outgoing from $\Lambda_x^{(1)}$ and for any $z \in \Lambda$, for which

$$\| \varphi(z) - x \| < \delta_x,$$

the inequality

$$t_z < \theta_x \| z \|$$

holds and the constants $\theta_x > 0$, $\delta_x > 0$ do not depend on $\epsilon > 0$.

Moreover for sufficiently small $\delta_x > 0$ and for any $\epsilon > 0$ there exist $N \in \mathbf{Z}_+$ and $\sigma > 0$ such that for any face Λoutgoing from $\Lambda_x^{(1)}$ and for any $z \in \Lambda$, for which

$$\| \varphi(z) - x \| < \delta_x,$$

the trajectory $\{T^t(z), 0 < t < t_z\}$ has at most N break points :

$$z_1 \text{ , } \dots \text{ , } z_k, \quad 0 \leq k \leq N,$$

with the property that for any $j = 1, ..., k$

$$\rho(\varphi(z_j), \mathfrak{C}^{(1)} \bigcup \mathfrak{C}^{(2)}) > \sigma.$$

Repeating now the arguments from the proof of lemma 3.2.4 and using lemmas 3.2.3 and 3.2.4 one can easily get (3.2.6).

Lemma 3.2.5 is proved.

For any $x \in \mathcal{X} \backslash \mathcal{X}_0$ let us fix the constants $\delta_x > 0$ and $\theta_x > 0$ from the lemma 3.2.5 and let us consider the sets

$$O_x = \{z \in I\!R_+ : z \neq 0, \| \varphi(z) - x \| < \delta_x\},$$

$$O_x^* = \{z \in I\!R_+ : z \neq 0, \| \varphi(z) - x \| < \frac{\delta_x}{2}\},$$

and

$$O_2 = \bigcup_{x \in \mathcal{X} \backslash \mathcal{X}_0} O_x \text{ , } O_2^* = \bigcup_{x \in \mathcal{X} \backslash \mathcal{X}_0} O_x \text{ .}$$

So for any $z \in (O_0 \bigcup O_1 \bigcup O_2 \bigcap) \mathbf{Z}_+^4$ the proposition 3.2.2 is proved. v To prove the proposition 3.2.2 for any other $x \in \mathbf{Z}_+^4$ we shall need also the following lemma.

Lemma 3.2.6 *Let Λ be either a nonergodic 1-face, which has no outgoing faces, or a nonergodic 2-face having two outgoing 3-faces. Then for any $\delta > 0$ and $\sigma > 0$ there exist $\delta_\Lambda > 0, \delta_\Lambda < \delta, \theta_\Lambda > 0$, and $c_\Lambda > 0$ such that*

$$P\{\exists\, t \in \mathbf{Z}_+ : 1 \leq t \leq \theta_\Lambda \parallel z \parallel,$$

$$\xi_z(t) \in O(\Lambda)\backslash\Lambda(1,2,3,4), \tag{3.2.7}$$

$$\delta_\Lambda \parallel \xi_z(t) \parallel < \rho(\xi_z(t),\Lambda) < \delta \parallel \xi_z(t) \parallel\} \geq 1 - \frac{c_\Lambda}{\parallel z \parallel}$$

for any $z \in \mathbf{Z}_+^4$ for which $\rho(z, \partial Q(\Lambda)) > \sigma \parallel z \parallel$, and $\rho(z,\Lambda) < \delta_\Lambda \parallel z \parallel$.

This lemma easily follows from the propositions 2.2.1 and 2.2.2.

Let us consider a nonergodic 2-face Λ having two outgoing 3-faces. Let $\Lambda^{(1)} \subset \overline{\Lambda}$ be a nonergodic 1-face, and $x \in \Lambda^{(1)} \bigcap \mathbf{Z}_+^4$. Then there are two possibilities : either $x \in \mathcal{X}$, or $x \notin \mathcal{X}$ and $\Lambda^{(1)}$ has a unique outgoing face. Repeating the arguments from the proof of the lemma 3.2.5 one can easily show that in the second case as well as in the first case there exist constants $\delta > 0$ and $\theta > 0$ and for any $\epsilon > 0$ there exists $c > 0$ such that for any $z \in \mathbf{Z}_+^4$, for which $\parallel \varphi(z) - x \parallel < \delta$,

$$P\{\tau_z^\epsilon > \theta \parallel z \parallel\} \leq \frac{c}{\parallel z \parallel}\,. \tag{3.2.8}$$

For an ergodic 1-face $\Lambda^{(1)} \subset \overline{\Lambda}$ by lemma 3.2.3 we have also (3.2.8) for any $z \in \mathbf{Z}_+^4, z \neq 0$, for which $\rho(z, \Lambda^{(1)}) < \delta_{\Lambda^{(1)}} \parallel z \parallel$.

From this and from the lemma 3.2.6 it easily follows that for any nonergodic 2-face Λ having two outgoing faces and for any $\delta > 0$ there exist $\delta_\Lambda > 0, \theta_\Lambda, c(\epsilon) > 0$ such that

$$P\{\text{Either } \tau_z^\epsilon \leq \theta_\Lambda \parallel z \parallel, \text{ or } \exists\, t = 1,\ldots,\theta_\Lambda \parallel z \parallel :$$

$$\xi_z(t) \in Q(\Lambda)\backslash\Lambda(1,2,3,4) \text{ and} \tag{3.2.9}$$

$$\delta_\Lambda \parallel \xi_z(t) \parallel < \rho(\xi_z(t),\Lambda) < \delta \parallel \xi_z(t) \parallel\} \geq 1 - \frac{c}{\parallel z \parallel}$$

for any $z \in \mathbf{Z}_+^4, z \neq 0$, for which $\rho(z,\Lambda) < \delta_\Lambda \parallel z \parallel$

Let us note now that for any $\epsilon > 0$ and for any nonergodic 2-face $\Lambda \subseteq \mathfrak{C}^{(2)}$ there exists $\delta > 0$ such that

$$t_z = \inf\{\text{ Either } \parallel T^{t_z}(z) \parallel < \frac{\epsilon}{2} \text{ or } T^{t_z}(z) \in O_0^* \bigcup O_1^* \bigcup O_2^*\} \leq$$

$$\leq \theta \parallel z \parallel$$

for any $z \in \mathbf{Z}_+^4, z \neq 0$, for which $\rho(z,\Lambda) < \delta \parallel z \parallel$, and for some positive constant $\theta > 0$.

Moreover there exist $N > 0$ and $\sigma > 0$ such that the trajectory $\{T^t(z), 0 < t < t_z\}$ has at most N break-points:

$$z_1, ..., z_k, \quad k \leq N,$$

and for any break point $z_j, \quad j = 1, ..., k,$

$$\rho(\varphi(z_j), \mathfrak{C}^{(1)} \bigcup \mathfrak{C}^{(2)}) > \sigma \, .$$

From this, using (3.2.12), proposition 2.3.1 and repeating the arguments from the proof of the lemma 3.2.2, we get that, for any nonergodic 2-face $\Lambda \subseteq \mathfrak{C}^{(2)}$, there exist $\delta_\Lambda > 0$ and $\theta_\Lambda > 0$ such that

$$P\{\tau_z^\epsilon > \theta_\Lambda \parallel z \parallel\} \leq \frac{c}{\parallel z \parallel} \tag{3.2.10}$$

for any $\epsilon > 0$ and for any $z \in \mathbf{Z}_+^4, z \neq 0$, having $\rho(z, \Lambda) < \delta_\Lambda \parallel z \parallel$, where $c = c(\epsilon) > 0$.

Let us fix for any nonergodic 2-face $\Lambda \subseteq C^{(2)}$ these constants $\delta_\Lambda > 0$ and $\theta_\Lambda > 0$ and consider the sets

$$\begin{aligned} O_\Lambda &= \{z \in I\!R_+^4 : z \neq 0 : \rho(z, \Lambda) < \delta_\Lambda \parallel z \parallel\} \\ O_\Lambda^* &= \{z \in I\!R_+^4 : z \neq 0 : \rho(z, \Lambda) < \frac{\delta_\Lambda}{2} \parallel z \parallel\}, \text{ and} \\ O_3 &= \bigcup_\Lambda O_\Lambda, O_3^* = \bigcup_\Lambda O_\Lambda^* \end{aligned} \tag{3.2.11}$$

where the union is over all 2-face $\Lambda \subseteq \mathfrak{C}^{(2)}$.

Let us now consider a nonergodic 1-face Λ, which has no outgoing faces. Repeating all the arguments which were used for a nonergodic 2-face having two outgoing faces and using the lemma 3.2.6 and the proposition 2.3.1 one can easily show that there exist $\delta_\Lambda > 0, \theta_\Lambda > 0$ and $c(\epsilon) > 0$ such that the estimate (3.2.10) holds for any $\epsilon > 0, z \in \mathbf{Z}_+^4, z \neq 0$, such that $\rho(z, \Lambda) < \delta_\Lambda \parallel z \parallel$, the estimate (3.2.10) holds.

Let us fix for any nonergodic 1-face Λ, which has no outgoing faces, these constants $\delta_\Lambda > 0$ and $\theta_\Lambda > 0$. Let us consider the sets (3.2.11) for this faces, and put

$$O_4 = \bigcup_\Lambda O_\Lambda \quad , \quad O_4^* = \bigcup_\Lambda O_\Lambda^*$$

where the union is over all nonergodic 1-faces which have no any outgoing face.

So for any $z \in (O_0 \bigcup ... \bigcup O_4) \bigcap \mathbf{Z}_+^4$ the proposition 3.2.2 is proved. To prove it for any other $z \in \mathbf{Z}_+^4, z \neq 0$, it is sufficient to note that for any $\epsilon > 0$ there exists $N \in \mathbf{Z}_+$ such that for any $z \in \mathbf{Z}_+^4 \setminus \bigcup_{j=0}^4 O_j, z \neq 0$, there exists $t_z \in I\!R_+$ such that :

(i) $t_z \leq \theta \parallel z \parallel$
for some positive constant $\theta > 0$,

(ii) either $\parallel T^{t_z}(z) \parallel < \frac{\epsilon}{2} \parallel z \parallel$, or $T^{t_z}(z) \in \bigcup_{j=0}^4 O_j^*$, and

(iii) the trajectory $\{T^t(z), 0 < t < t_z\}$ has at most N break-points.

After this, using the proposition 2.3.1, one can easily get the proposition 3.2.2

Proposition 3.2.2 is proved

Using the proposition 3.2.2 let us now prove the ergodicity of the chain $\mathfrak{L}$.

Let $\epsilon > 0, \epsilon < 1$ be fixed, and let $\theta > 0$ and $c > 0$ be the constants from the proposition 3.2.2. For any $z \in \mathbf{Z}_+^4$, we shall define the stopping time $\tau_1(z)$ of the r.w. $\xi_z(t)$ by setting

$$\tau_1(z) = \min\{\tau_z^\epsilon, [\theta \parallel z \parallel]\}, \text{ if } \parallel z \parallel \geq \frac{1}{\theta}, \text{ and}$$

$$\tau_1(z) = 1 \quad \text{otherwise}$$

Then from the proposition 3.2.2 by boundedness of jumps condition for the r.w. $\xi_z(t)$ for any $z \in \mathbf{Z}_+^4$, for which $\parallel z \parallel \geq \frac{1}{\theta}$, we get

$$E \parallel \xi_z(\tau_1(z)) \parallel \leq \epsilon \parallel z \parallel + (1 + d\theta)c.$$

From this it easily follows that there exist $\tilde{\epsilon} > 0$ and $\mathcal{D} > \frac{1}{\theta}$ such that for any $z \in \mathbf{Z}_+^4$, for which $\parallel z \parallel > \mathcal{D}$, the following estimate holds

$$E \parallel \xi_z(\tau_1(z)) \parallel \leq (1 - \tilde{\epsilon}) \parallel z \parallel \tag{3.2.12}$$

Note now that for any $z \in \mathbf{Z}_+^4$, for which $\parallel z \parallel > \frac{1}{\theta}$,

$$E\tau_1(z) \leq \theta \parallel z \parallel \tag{3.2.13}$$

From (3.2.12) and (3.2.13) it follows that for any $z \in \mathbf{Z}_+^4$, for which $\parallel z \parallel > \mathcal{D}$,

$$E \parallel \xi_z(\tau_1(x)) \parallel \leq \parallel z \parallel - \frac{\tilde{\epsilon}}{\theta} E\tau_1(z) \tag{3.2.14}$$

Let us define the sequence of stopping-times $\tau_n(z)$ of the r.w. $\xi_z(t)$ by setting $\tau_0(z) \equiv 0$ and for any $n \in \mathbf{Z}_+ \tau_{n+1}(z) = \tau_n(z) + \tau_1(\xi_z(\tau_n(z)))$.

Then by (3.2.14) the conditions of the proposition 3.2.1 are satified and consequently the chain $\mathfrak{L}$ is ergodic.

So the theorem 1.2.5 is proved.

Chapter 4

Essential scattering

4.1 Nonergodicity

Let us consider the associated graph $\mathcal{G}$, the set of vertices of this graph $\mathcal{X}$, and

$$W_+ = \bigcup_{x\in\mathcal{X}} W_+(x)\ ,\ W_- \bigcup_{x\in\mathcal{X}} W_-(x)$$

where $W_+(x)$ $(W_-(x))$ is the set of all separatrices outgoing from the point x (ingoing to the point x).

Let there exist $\gamma_0 \in W_+ \bigcap W_-$ for which

$$\mathcal{M}_{\gamma_0} > 0 \tag{4.1.1}$$

(see proposition 1.2.7).
Let us show that in this case the r.w. $\mathfrak{L}$ is nonergodic. We shall do it for the case when the associated Markov chain $\mathfrak{A}$ is irreducible (in particular $W_+ = W_- = W$) . For any other cases one can easily do it analogously.

By the theorem 1.2.1 it is sufficient to consider the case when for any ergodic face Λ the vector V_Λ has at least one negative component.

So we shall always assume here that for any ergodic face Λ the vector V_Λ has at least one negative component.

Let us consider for any $\gamma \in W$ the point $x(\gamma) \in \mathcal{X}$, such that $\gamma \in W_-(x(\gamma))$, the 1-face $\Lambda_\gamma^{(1)}$ containing the point x, and the ergodic 2-face $\Lambda_\gamma^{(2)}$ ingoing to the face $\Lambda_\gamma^{(1)}$, along which the separatrix γ goes to $x(\gamma)$. (This 2-face $\Lambda_\gamma^{(2)}$ exists by the assumption A_{10}). Let us consider also for any $\gamma \in W$ the Markov chain $\mathfrak{L}_{\Lambda_\gamma^{(1)},\Lambda_\gamma^{(2)}}$ having the state of space

$$C^0_{\Lambda_\gamma^{(2)}} \bigcap \mathbf{Z}^4_+ = (C_{\Lambda_\gamma^{(1)}} \backslash Q(\Lambda_\gamma^{(2)})) \bigcap \mathbf{Z}^4_+\ ,$$

and transition probabilities $p_{\Lambda_\gamma^{(1)},\Lambda_\gamma^{(2)}}(x,y)$, $x,y \in C^0_{\Lambda_\gamma^{(2)}} \bigcap \mathbf{Z}^4_+$ (see proof of the proposition 1.5.2).

We shall prove all the theorems 1.2.8 -1.2.11 for the case when the Markov chains $\mathfrak{L}_{\Lambda_\gamma^{(1)},\Lambda_\gamma^{(2)}}$ is ergodic. In other cases one can prove this theorems similarly. Let $\pi_{\Lambda_\gamma^{(1)},\Lambda_\gamma^{(2)}}(x)$, $x \in C^0_{\Lambda_\gamma^{(2)}} \bigcap \mathbf{Z}^4_+$ be stationary probabilities of this chain.

Let us consider for any $\epsilon > 0$ and for any $\gamma \in W$ the set

$$Y_\gamma^\epsilon = \{z \in Q(\Lambda_\gamma^{(1)}) \backslash Q(\Lambda_\gamma^{(2)}) : z \neq 0, \| \varphi(z) - x(\gamma) \| < \epsilon\}$$

and for any $z \in \mathbf{Z}_+^4 \backslash \{0\}$, for which $\| \varphi(z) - x(\gamma) \| < \epsilon$, let us consider the r.w. $\xi_z(t)$ corresponding to the chain $\mathfrak{L}$ starting at the point z, the first time $\tau_z(\partial Q(\Lambda_\gamma^{(1)}))$ when the r.w. $\xi_z(t)$ hits the set $\partial Q(\Lambda_\gamma^{(1)}) = \overline{Q(\Lambda_\gamma^{(1)})} \backslash Q(\Lambda_\gamma^{(1)})$, and the next after $\tau_z(\partial Q(\Lambda_\gamma^{(1)})) - 1$ time τ_z^ϵ when the r.w. $\xi_z(t)$ hits the set $\bigcup_{\gamma' \in W} Y_{\gamma'}^\epsilon \bigcup \{0\}$.

Proposition 4.1.1 *For any $\delta > 0$ there exist $\epsilon > 0, c > 0$ and $\theta_1 > 0, \theta_2 > 0$ such that for any $\gamma \in W$ and for any $z \in \mathbf{Z}_+^4 \backslash \{0\}$ for which $\| \varphi(z) - x(\gamma) \| < \epsilon$,*

$$\text{(i)} \sum_{\gamma' \in W_+(x(\gamma))} | P\{\theta_1 \| z \| < \tau_z^\epsilon < \theta_2 \| z \|, \ \xi_z(\tau_z^\epsilon) \in Y_{\gamma'}^\epsilon,$$
$$| \, \| \xi_z(\tau_z^\epsilon) \| - K_{\gamma'} \| z \| \, | < \delta \| z \| \} - g_{\Lambda_\gamma^{(1)}}(z, \Lambda_{\gamma'}) | < \frac{c}{\| z \|}, \tag{4.1.2}$$

where $\Lambda_{\gamma'}$ is a 2-face outgoing from $\Lambda_{x(\gamma)}^{(1)}$, along which the separatrix γ' goes from the point $x(\gamma)$; $g_{\Lambda_\gamma^{(1)}}(z, \Lambda_{\gamma'})$ is the probability that the induced chain $\mathfrak{L}_{\Lambda_\gamma^{(1)}}$ goes to infinity from the point z along the face $\Lambda_{\gamma'}$ (see § 1.5). Moreover

$$\text{(ii)} \sum_{\gamma' \in W_+(x(\gamma))} \ \sum_{y \in C^0_{\Lambda_{\gamma'}^{(2)}} \bigcap \mathbf{Z}_+^4} \Big| \sum_{y' \in Y_{\gamma'}^\epsilon \bigcap \mathbf{Z}_+^4}^{(y)} P\{\xi_z(\tau_z^\epsilon) = y'\} - g_{\Lambda_\gamma^{(1)}}(z, \Lambda_{\gamma'}) \pi_{\gamma'}(y) \Big| \to 0$$

$$as \ \| z \| \to 0 \, .$$

where the summation $\sum^{(y)}$ is over all $y' \in Y_{\gamma'}^\epsilon \bigcap \mathbf{Z}_+^4$ for which the orthogonal projection onto $C^0_{\Lambda_\gamma^{(2)}}$ is y.

One can easily get this proposition from the propositions 2.1.8, 2.2.2 and 2.3.1 and from ergodicity of the chain $\mathcal{L}_{\Lambda_\gamma^{(1)}, \Lambda_\gamma^{(2)}}$ using the same arguments as in the proof of lemma 3.1.4.

Let us note now that for irreducible associated chain $\mathfrak{A}$ the value $\mathcal{M}_\gamma$ does not depend on $\gamma \in W$. By the proposition 1.2.7 it follows that there exist $\delta > 0, N^* \in \mathbf{Z}_+$ and $\mathcal{M}^* > 0$ such that for any $N \geq N^*, N \in \mathbf{Z}_+$, and for any $\gamma_1 \in W$

$$\log\{ \sum_{\gamma_2, \ldots, \gamma_N \in W_+ \bigcap W_-} p_{\mathfrak{A}}(\gamma_1, \gamma_2) \ldots p_{\mathfrak{A}}(\gamma_{N-1}, \gamma_N) \prod_{j=2}^{N} (K_{\gamma_j} - \delta)\} > \mathcal{M}^* N \tag{4.1.3}$$

Let these constants $\delta > 0, N^* \in \mathbf{Z}_+$ and $\mathcal{M}^* > 0$ be fixed.

For given $\delta > 0$ let us fix the constant $\epsilon > 0$ from the proposition 4.1.1 and let us define for any $\gamma \in W_+ \bigcap W_-$ and for any $z \in Y_\gamma^\epsilon \bigcap \mathbf{Z}_+^4$ the sequence of stopping-times $\tau_n(z)$ of the r.w. $\xi_z(t)$, by setting $\tau_0(z) = 0, \tau_0(\underline{0}) = \tau_1(\underline{0}) = 0, \tau_1(z) = \tau_z^\epsilon$, and for any $n \in \mathbf{Z}_+$

$$\tau_{n+1}(z) = \tau_n(z) + \tau_1(\xi_z(\tau_n(z))) \ .$$

It is easy to see that for any $\gamma \in W$, for any $z \in Y_\gamma^\epsilon \bigcap \mathbf{Z}_+^4$ and for any $N \geq N^*, N \in \mathbf{Z}_+$,

$$\begin{aligned} E \parallel \xi_z(\tau_N(z)) \parallel \geq \parallel z \parallel \sum_{\gamma_1 \dots \gamma_N \in W} \prod_{j=1}^{N} (K_{\gamma_j} - \delta) \times \\ \times P\{\forall k = 1, \dots, N : \xi_z(\tau_k(z)) \in Y_{\gamma_k}^\epsilon \text{ and} \\ \parallel \xi_z(\tau_k(z)) \parallel \geq (K_{\gamma_k} - \delta) \parallel \xi_z(\tau_{k-1}(z)) \parallel\} \end{aligned} \tag{4.1.4}$$

From the proposition 4.1.1 it easily follows that for any $N \geq N^*, N \in \mathbf{Z}_+$, and for any $\gamma, \gamma_1 \in W$

$$\begin{aligned} \sum_{\gamma_2, \dots, \gamma_N \in W} \mid P\{\forall k = 1, \dots, N : \xi_z(\tau_k(z)) \in Y_{\gamma_k}^\epsilon \ , \text{ and} \\ \parallel \xi_z(\tau_k(z)) \parallel > (K_{\gamma_k} - \delta) \parallel \xi_z(\tau_{k-1}(z)) \parallel\} - \\ - g_{\Lambda_\gamma^{(1)}}(z, \Lambda_{\gamma_1}) p_{\mathfrak{A}}(\gamma_1, \gamma_2) \dots p_{\mathfrak{A}}(\gamma_{N-1}, \gamma_N) \mid \to 0 \end{aligned} \tag{4.1.5}$$

as $\parallel z \parallel \to \infty, z \in Y_\gamma^\epsilon \bigcap \mathbf{Z}_+^4$.

From (4.1.4), (4.1.5) it follows that for any $N \geq N^*$ and for any $\sigma > 0$ there exists $\mathcal{D} > 0$ such that

$$\begin{aligned} E \parallel \xi_z(\tau_N(z)) \parallel \quad \geq \quad \parallel z \parallel \{ \sum_{\gamma_1 \dots \gamma_N \in W} \prod_{j=1}^{N} (K_{\gamma_j} - \delta) \times \\ g_{\Lambda_\gamma^{(1)}}(z, \Lambda_{\gamma_1}) p_{\mathfrak{A}}(\gamma_1, \gamma_2) \dots p_{\mathfrak{A}}(\gamma_{N-1}, \gamma_N) - \sigma K^N \} \end{aligned} \tag{4.1.6}$$

for any $\gamma \in W$ and for any $z \in Y_\gamma^\epsilon \bigcap \mathbf{Z}_+^4$, for which $\parallel z \parallel > \mathcal{D}$ where $K = \max_{\gamma \in W} K_\gamma$.

From (4.1.6) by (4.1.3) we get

$$E \parallel \xi_z(\tau_N(z)) \parallel \geq \parallel z \parallel \{e^{\mathcal{M}^* N} - \sigma K^N\} \tag{4.1.7}$$

Let us choose $N \geq N^*$ and $\sigma > 0$, such that

$$\omega = e^{\mathcal{M}^* N} - \sigma K^N > 1 \ ,$$

and for given N and σ let us fix $\mathcal{D} > 0$ from (4.1.6).

Let us consider the following function $f : \mathbf{Z}_+^4 \to I\!R_+$:

$$f(z) = 0 \ , \text{ if } \parallel z \parallel \leq \mathcal{D}, \text{ and}$$
$$f(z) = \parallel z \parallel \ , \text{ if } \parallel z \parallel > \mathcal{D} \ .$$

Then by (4.1.7) for any $z \in \{ \bigcup_{\gamma \in W} Y_\gamma^c \bigcup \{\underline{0}\} \} \bigcap \mathbf{Z}_+^4$ we have

$$E\ f(\xi_z(\tau_N(z))) \geq \omega f(z). \tag{4.1.8}$$

Let us note now that by the proposition 4.1.1 for any $z \in (\bigcup_{\gamma \in W} Y_\gamma^c \bigcup \{0\}) \bigcap \mathbf{Z}_+^4$

$$P\{\tau_N(z) \geq \theta_1 \parallel z \parallel\} \geq 1 - \frac{c}{\parallel z \parallel},$$

and consequently

$$E\tau_N(z) \geq \theta_1 f(z) - c\theta_1 \tag{4.1.9}$$

From (4.1.8) and (4.1.9) for any $z \in (\bigcup_{\gamma \in W} Y_\gamma^c) \bigcap \mathbf{Z}_+^4$ and for any $k \in \mathbf{Z}_+$ we get

$$E(\tau_{(k+1)N}(z) - \tau_{kN}(z)) \geq \theta_1 Ef(\xi_z(\tau_{kN}(z))) - c\theta_1 \geq \theta_1 \omega^k f(z) - c\theta_1 \tag{4.1.10}$$

By (4.1.10) for any $z \in \bigcup_{\gamma \in W} Y_\gamma^c \bigcap \mathbf{Z}_+^4$, for which $\parallel z \parallel > \mathcal{D}$,

$$E\tau_{kN}(z) \to \infty \text{ as } k \to \infty \tag{4.1.11}$$

Let $z \in \bigcup_{\gamma \in W} Y_\gamma^c \bigcap \mathbf{Z}_+^4, \parallel z \parallel > \mathcal{D}$, and let τ_z be the first time when the r.w. $\xi_z(t)$ the point $\underline{0}$. Then by definition for any $k \in \mathbf{Z}_+$

$$\tau_z \geq \tau_{kN}(z). \tag{4.1.12}$$

From (4.1.11) and (4.1.12) we get that

$$E\tau_z = \infty.$$

So the chain $\mathfrak{L}$ is nonergodic.

Theorem 1.2.8 is proved.

4.2 Ergodicity

Let for any ergodic face Λ the vector V_Λ have at least one negative component, and let the associated graph $\mathcal{G}$ have a cycle. Note that in this case the semigroup $T_S^t, t \in \mathbb{R}_+$ has no periodic trajectories.

Let

$$\mathcal{M}_{\mathfrak{A}} = \max_\gamma \mathcal{M}_\gamma < 0 \tag{4.2.1}$$

Let us show that in this case the r.w. $\mathfrak{L}$ is ergodic. We shall do it for the case when the associated chain $\mathfrak{A}$ is irreducible (in particular $W_+ = W_- = W$). In any other case one can easily do it analogously.

Let us fix constants $N^* \in \mathbf{Z}_+, \delta > 0$ and $\mathcal{M}^* > 0$ such that for any $N \geq N^*$, $N \in \mathbf{Z}_+$, and for any $\gamma \in W$.

$$\log\{ \sum_{\gamma_2,\ldots,\gamma_N \in W} p_{\mathfrak{A}}(\gamma_1,\gamma_2)\ldots p_{\mathfrak{A}}(\gamma_{N-1},\gamma_N) \prod_{j=1}^{N}(K_{\gamma_j}+\delta)\} < -\mathcal{M}^* N. \tag{4.2.2}$$

Due to (4.2.1), it is always possible to satisfy (4.2.2) since proposition 1.2.7 holds.

For given $\delta > 0$ let us fix the constants $\epsilon > 0, \epsilon < 1, \theta_1 > 0$ and $\theta_2 > 0$ from the proposition 4.1.1.

We shall use here notations, introduced in §4.1.

Let us note that for any $z \in \mathbb{R}^4_+$ there exists $t \in \mathbb{R}_+$ such that

$$T^t(z) \in \bigcup_{\gamma \in W} Y_\gamma^{\frac{\epsilon}{2}} \bigcup \{0\} .$$

Let us consider for any $z \in \mathbb{R}^4_+$

$$t_z = \inf \{t \in \mathbb{R}_+ \ : \ \text{either } T^t(z) \in \textstyle\bigcup_{\gamma \in W} Y_\gamma^{\frac{\epsilon}{2}} \bigcup\{0\} \text{or } \parallel T^t(z) \parallel < \frac{\epsilon}{2} \parallel z \parallel\}$$

It is easy to see that for any $z \in \mathbb{R}^4_+$

$$t_z \leq \theta \parallel z \parallel \tag{4.2.3}$$

where the constant $\theta > 0$ does not depend on z

Let us consider starting at the point $z \in \mathbf{Z}^4_+$ the r.w. $\xi_z(t)$ corresponding to the chain $\mathfrak{L}$. Let us define a stopping-time $\tau_1(z)$ of the r.w. $\xi_z(t)$ by setting $\tau_1(\underline{0}) = 1$,

$$\tau_1(z) = \min \{\tau_z^\epsilon, \theta_2 \parallel z \parallel 1\}, \text{ if } \parallel \varphi(z) - x(\gamma) \parallel < \epsilon \text{ for some } \gamma \in W \text{ and } \parallel z \parallel > \frac{1}{\theta},$$

$$\tau_1(z) = t_z, \ if \parallel \varphi(z) \parallel \leq \epsilon \text{ for all } \gamma \in W, z \neq \underline{0},$$

and

$$\tau_1(z) = 1 \quad \text{otherwise.}$$

Here τ_z^ϵ is the stopping-time of the r.w. $\xi_z(t)$ which was introduced in §4.1, and $\theta_2 > 0$ is the constant from the proposition 4.1.1.

Let $\tau_n(z), n \in \mathbf{Z}_+$, be the sequence of stopping-times of the r.w. $\xi_z(t)$, such that for all $z \in \mathbf{Z}^4_+$

$$\tau_0(z) = 0 \ \text{ and for any } \ n \in \mathbf{Z}_+$$

$$\tau_{n+1}(z) = \tau_n(z) + \tau_1(\xi_z(\tau_n(z))) .$$

Proposition 4.2.1 *There exist $\omega > 0, \omega < 1, N \in \mathbf{Z}_+$ and $\mathcal{D} > 0$ such that for any $z \in \mathbf{Z}_+^4$, for which $\| z \| > \mathcal{D}$,*

$$E \| \xi_z(\tau_N(z)) \| < \omega \| z \| \ . \tag{4.2.4}$$

Proof.

Using the arguments from the proof of the theorem 1.2.5 one can easily show that for any $z \in \mathbf{Z}_+^4 \setminus \{0\}$ for which $\| \varphi(z) - x(\gamma) \| > \epsilon$ for all $\gamma \in W$, the following estimate holds

$$\begin{gathered} P\{\text{either } \ \| \xi_z(\tau_1(z)) \| \leq \epsilon \| z \|, \text{ or } \exists \gamma \in W : \\ \| \varphi(\xi_z(\tau_1(z))) - x(\gamma) \| < \epsilon\} \geq 1 - \tfrac{c}{\|z\|} \end{gathered} \tag{4.2.5}$$

where the constant $c = c(\epsilon) > 0$ does not depend on z.

Note also that due to (4.2.3) for any $z \in \mathbf{Z}_+^4 \setminus \{0\}$, such that $\| \varphi(z) - x(\gamma) \| > \epsilon$ for all $\gamma \in W$,

$$\| \xi_z(\tau_1(x)) \| \leq \| z \| (1 + \theta d) \quad \text{a.s.} \tag{4.2.6}$$

From (4.2.5) and (4.2.6) it easily follows that to prove the proposition 4.2.1 it is sufficient to show that there exist $\omega > 0, \omega < \frac{1}{1+\theta d}, N \in \mathbf{Z}_+$ and $\mathcal{D} > 0$ such that for any $\gamma \in W$ and for any $z \in \mathbf{Z}_+^4$, for which $\| z \| > \mathcal{D}$ and $\| \varphi(z) - x(\gamma) \| < \epsilon$, the inequality (4.2.4) holds.

To show it let us note that from the proposition 4.1.1 it easily follows that for any $\gamma \in W$, for any $z \in \mathbf{Z}_+^4 \setminus \{0\}$, for which $\| \varphi(z) - x(\gamma) \| < \epsilon$, and for any $N \in \mathbf{Z}_+$

$$E \| \xi_z(\tau_N(z)) \| \leq \| z \| \sum_{\gamma_1, \ldots, \gamma_N \in W} \prod_{j=1}^{N} (K_{\gamma_j} - \delta) \times$$

$$\times P\{\forall k = 1, \ldots, N : \xi_z(\tau_k(z)) \in Y_{\gamma k}^{\epsilon} \ , \tag{4.2.7}$$

$$(K_{\gamma_k} - \delta) \| \xi_z(\tau_{k-1}(z)) \| \leq \| \xi_z(\tau_k(z)) \| \leq K_{\gamma_k} + \delta) \| \xi_z(\tau_{k-1}(z)) \| \} + c$$

where the constant $c = c(\epsilon, \delta, N) > 0$ does not depend on z.

From (4.2.7) using the arguments from §4.1 one can easily get that for any $\sigma > 0$ and $N \geq N^*$ there exists $\mathcal{D} > 0$ such that for any $\gamma \in W$ and for any $z \in \mathbf{Z}_+^4$ for which $\| \varphi(z) - x(\gamma) \| < \epsilon$ and $\| z \| > \mathcal{D}$,

$$E \| \xi_z(\tau_N(z)) | \leq \| z \| (e^{-\mathcal{M}^* N} + \sigma(K + \delta)^N) + c \tag{4.2.8}$$

where $K = \sup_\gamma K_\gamma$.

Let us choose $N \geq N^*, N \in \mathbf{Z}_+, \sigma > 0$ and $\omega > 0$ such that

$$e^{-\mathcal{M}^* N} + \sigma(K + \delta)^N < \omega < \frac{1}{1 + \theta d}$$

Then, due to (4.2.8), there exists $\mathcal{D} > 0$, such that for any $\gamma \in W$ and for any $z \in \mathbf{Z}_+^4$, for which $\| z \| > \mathcal{D}$ and $\| \varphi(z) - x(\gamma) \| < \epsilon$, the inequality (4.2.4) holds.

Proposition 4.2.1 is proved.

Using the proposition 4.2.1 one can easily prove ergodicity of the chain $\mathfrak{L}$ similarly to the proof of the theorem 1.2.5 (see §3.2).

4.3 Transience

Let there exist an irreducible class W_j of essential states of the associated Markov chain $\mathfrak{A}$ for which

$$\mathfrak{L}_{W_j} > 0 \tag{4.3.1}$$

Let us show that in this case the r.w. $\mathfrak{L}$ is transient. We shall do it for the case when the associated chain $\mathfrak{A}$ is irreducible ($W_j = W$). In any other cases one can easily do it analogously.

We shall use here the notations introduced in §4.1.

Let us fix constants $N^* \in \mathbf{Z}_+, \delta > 0$ and $L^* > 0$ such that for any $N \geq N^*, N \in \mathbf{Z}_+$, and for any $\gamma_1 \in W$

$$\sum_{\gamma_2 \ldots \gamma_n \in W} p_{\mathfrak{A}}(\gamma_1, \gamma_2) \ldots p_{\mathfrak{A}}(\gamma_{N-1}, \gamma_N) \log\{\prod_{j=1}^{N}(K_{\gamma_j} - \delta)\} > L^* N. \tag{4.3.2}$$

Due to (4.3.1) it is always possible to satisfy (4.3.2) since for any $\gamma, \gamma' \in W$

$$\frac{1}{n} \sum_{k=1}^{n} p_{\mathfrak{A}}^{(k)}(\gamma, \gamma') \to \pi_{\mathfrak{A}}(\gamma') \text{ as } n \to \infty, \tag{4.3.3}$$

where $\pi_{\mathfrak{A}}(\gamma), \gamma \in W$, are the stationary probabilities of the chain $\mathfrak{A}$.

For given $\delta > 0$ let us fix the contant $\epsilon > 0$ from the proposition 4.1.1.

For any $z \in (\bigcup_{\gamma \in W} Y_\gamma^\epsilon \bigcup \{\underline{0}\}) \bigcap \mathbf{Z}_+^4$ let us consider a stopping-time τ_z^ϵ which was introduced in §4.1, and let us consider a sequence of stopping-times $\tau_n(z)$, where $\tau_0 = 0$, $\tau_1(z) = 1$, $\tau_1(z) = \tau_z^\epsilon$ if $z \neq 0$, and for any $n \in \mathbf{Z}_+$

$$\tau_{n+1}(z) = \tau_n(z) + \tau_1(\xi_z(\tau_n(z)))$$

To prove transience of the chain $\mathfrak{L}$ it is sufficient to consider the case when for any $z \in (\bigcup_{\gamma \in W} Y_\gamma^\epsilon \bigcup \{\underline{0}\}) \bigcap \mathbf{Z}_+^4$ and for any $n \in \mathbf{Z}_+$

$$\tau_n(z) < \infty \text{ a.s.}$$

Let us consider for any $z \in (\bigcup_{\gamma \in W} Y_\gamma^\epsilon \bigcup \{\underline{0}\}) \bigcap \mathbf{Z}_+^4$ a sequence

$$\zeta_z(n) = \xi_z(\tau_{nN^*}(z)), n \in \mathbf{Z}_+.$$

To prove transience of the chain $\xi_z(t)$ it is sufficient to prove transience of the chain $\zeta_z(n)$. To prove transience of the chain $\zeta_z(n)$ we shall use the proposition 3.1.2.

Let us consider on the state space of $\zeta_z(n)$ the following function f :

$$f(z) = \frac{1}{\log \|z\|} \text{ if } \| z \| > 3, \text{ and}$$
$$f(z) = 1 \text{ otherwise .}$$

Proposition 4.3.1 *There exists $\mathcal{D} > 0$ such that for any $\gamma \in W$ and for any $z \in Y_\gamma^\epsilon \bigcap \mathbf{Z}_+^4$, for which $\| z \| > \mathcal{D}$,*

$$E\ f(\zeta_z(1)) \leq f(z) \tag{4.3.4}$$

Proof.

From the proposition 4.1.1 it easily follows that for any $\gamma \in W$ and for any $z \in Y_\gamma^\epsilon \bigcap \mathbf{Z}_+^4$ the following estimate holds

$$E\ f(\zeta_z(1)) = E\ f(\xi_z(\tau_{N^*}(z))) \leq$$

$$\leq \sum_{\gamma_1,\ldots,\gamma_{N^*} \in W} (\log(\| z \| \prod_{j=1}^{N} (K_{\gamma_j} - \delta)))^{-1}.P\{\forall k = 1,\ldots,N : \tag{4.3.5}$$

$$\xi_z(\tau_k(z)) \in Y_{\gamma_k}^\epsilon, \text{ and } \| \xi_z(\tau_k(z)) \| > (K_{\gamma_{k-1}} - \delta) \| \xi_z(\tau_{k-1}(z)) \|\} + \frac{c}{\|z\|}$$

where the constant $c > 0$ does not depend on z.

Note now that for sufficiently large $\| z \|$

$$\begin{aligned} &(\log \ \| z \| \textstyle\prod_{j=1}^{N}(K_{\gamma j} - \delta))^{-1} = \\ &XS \\ &= \frac{1}{\log \|z\|} - \frac{\log (\prod_{j=1}^{N}(K_{\gamma_j} - \delta))}{(\log\|z\|)^2} + \\ &+ o((\log \ \| z \|)^{-2}) \end{aligned} \tag{4.3.6}$$

From (4.3.5) and (4.36) we get

$$E\ f(\xi_z(\tau_{N^*}(z))) \leq \frac{1}{\log\|z\|} + o((\log \| z \|)^{-2}) -$$

$$- \sum_{\gamma_1,\ldots,\gamma_{N^*} \in W} (\log \| z \|)^{-2} \log(\prod_{j=1}^{N} (K_{\gamma_j} - \delta)) \times \tag{4.3.7}$$

$$\times P\{\forall k = 1,\ldots,N : \xi_z(\tau_k(z)) \in Y_{\gamma_k}^\epsilon, \text{ and}$$

$$\| \xi_z(\tau_k(z)) \| > (K_{\gamma_k} - \delta) \| \xi_z(\tau_{k-1}(z)) \|\}$$

Due to (4.3.2) using the same arguments as in the proof of the theorem 1.2.8 (see §4.1) one can easily show that for any $\sigma > 0$ there exists $\mathcal{D} > 0$, such that for any $\gamma \in W$ and for any $x \in Y_\gamma^\epsilon \bigcap \mathbf{Z}_+^4$, for which $\| z \| > \mathcal{D}$,

$$\sum_{\gamma_1,\dots,\gamma_{N^*}\in W} \log \prod_{j=1}^{N} (K_{\gamma_j} - \delta) P\{\forall k = 1,\dots,N : \xi_z(\tau_k) \in Y_{\gamma_k}^{\epsilon}$$
$$\| \xi_z(\tau_k(z)) \| > (K_{\gamma_k} - \delta) \| \xi_z(\tau_{k-1}(z)) \| \} > L^* N^* - \sigma N^* \log K \tag{4.3.8}$$

where $K = \max_\gamma K_\gamma$.

Let us choose $\sigma > 0$ such that

$$\omega = L^* - \sigma N^* \log K > 0 .$$

Then from (4.3.7) and (4.3.8) for any $\gamma \in W$ and for any $z \in Y_\gamma^\epsilon \bigcap \mathbf{Z}_+^4$, for which $\| z \| > \mathcal{D}$, we get

$$E\, f(\xi_z(\tau_{N^*}(z))) \leq \frac{1}{\log \| z \|} - \frac{\omega}{(\log \| z \|)^2} + o(\frac{1}{(\log \| z \|)^2})$$

From this (4.3.4) easily follows.

Proposition 4.3.1 is proved.

Due to the propositions 4.3.1 and 3.1.2 the chain $\zeta_z(n)$ is transient, and consequently the chain $\xi_z(t)$ is also transient.

Theorem 1.2.10 is proved.

4.4 Recurrence

Let for any ergodic face Λ the vector V_Λ have at least one negative component, and let the associated graph $\mathcal{G}$ have a cycle.

Since in this case the semigroup $T_S^t, t \in I\!R_+$, has no periodic trajectories, then the conditions of the theorem 1.2.4 are not satisfied.

Let for any irreducible class of essential states W^j of the associated chain $\mathfrak{A}$

$$\mathfrak{L}_{W^j} < 0 . \tag{4.4.1}$$

Let us show that in this case the r.w. $\mathfrak{L}$ is recurrent. We shall do it for the case when the associated chain $\mathfrak{A}$ is irreducible ($W_+ = W$ is a unique irreducible class). In other cases one can easily do it by analogously.

Let us choose constants $N^* \in \mathbf{Z}_+, \delta > 0$ and $L^* > 0$, such that for any $N \geq N^*$, $N \in \mathbf{Z}_+$ and for any $\gamma_1 \in W$

$$\sum_{\gamma_2,\dots,\gamma_N\in W} p_{\mathfrak{A}}(\gamma_1,\gamma_2)\dots p_{\mathfrak{A}}(\gamma_{N-1},\gamma_N) \log \prod_{j=2}^{N} (K_{\gamma_j} + \delta) < -L^* N \tag{4.4.2}$$

Due to (4.4.1) it is always possible to satisfy (4.4.2) since (4.3.3) holds.

For given $\delta > 0$ let us fix the constants $\epsilon > 0, \theta_1 > 0$ and $\theta_2 > 0$ from the proposition 4.1.1.

Let us consider the r.w. $\xi_z(t)$ starting at the point $z \in \mathbf{Z}_+^4$ corresponding to the chain $\mathfrak{L}$, and let us consider the sequence of stopping-times $\tau_n(z), n \in \mathbf{Z}_+$, of the r.w. $\xi_z(t)$, which had been introduced in §4.2.

Proposition 4.4.1 *There exist* $\omega > 0, n \in \mathbf{Z}_+$ *and* $\mathcal{D} > 0$, *such that for any* $z \in \mathbf{Z}_+^4$, *for which* $\| z \| > \mathcal{D}$

$$E \; log \; \| \xi_z(\tau_N(z)) \| \leq \; log \; \| z \| - \omega \; .$$

One can easily prove this proposition using the same arguments as in the proof of the proposition 4.2.1, and using (4.4.2) instead of (4.2.2). From the proposition 4.4.1 recurrence of the chain $\zeta_z(n) = \xi_z(\tau_{nN}(z)), n \in \mathbf{Z}_+$ follows, by the well known recurrence criteria [6]: a Markov chain with the state space $\mathbf{Z}_+$ and transition probabilities $p_{ij}, i, j \in \mathbf{Z}_+$, is recurrent if there exists a nonnegative function f on $\mathbf{Z}_+$ an a finite set $A \subset \mathbf{Z}_+$, such that for any $i \in \mathbf{Z}_+ \backslash A$

$$\sum_{j \in \mathbf{Z}_+} p_{ij} \; f_j - f_i \leq 0 \; ,$$

and $f_i \rightarrow$ as $j \rightarrow \infty$.

From recurrence of the chain $\zeta_z(n)$ recurrence of the r.w. $\xi_z(t)$ follows.

So the r.w. $\mathfrak{L}$ is recurrent. Theorem 1.2.11 is proved.

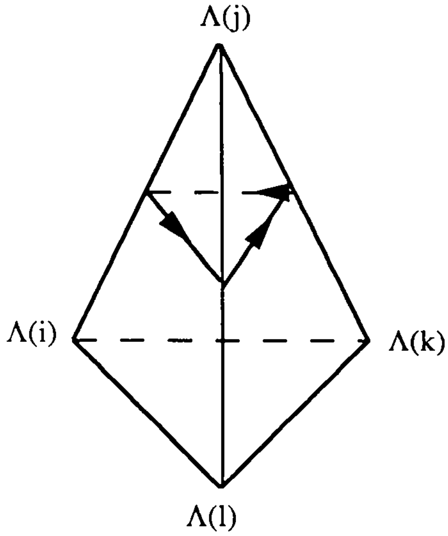

Fig. 1.4.1.

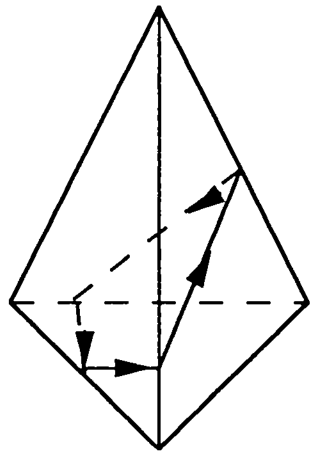

Fig. 1.4.2.

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